Local error estimator for stresses in 3D structural analysis

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Abstract

This paper focuses on an a posteriori error estimator for FE approximations of 3D linear elasticity problems. The objective is to present the application of the new generation of error in constitutive relation to the calculation of the local error in classical tetrahedral elements. We show on examples whose solution is known analytically that the local error estimation gives satisfactory effectivity indexes.

Keywords: A posteriori error estimation; Local error; Stress

1. Introduction

Today, the use of numerical simulations in mechanical design is very widespread. Complex finite element calculations are employed in a routine and daily way. Quality control of the results provided by finite element calculations has been a major concern for many years, as has been the case in industry and in research.

Methods have been developed over many years to evaluate the global quality of finite element analyses [1–3]. For linear problems, all these methods provide a global energy-based estimate of the discretization error. Most of the time, such global information is totally insufficient for dimensioning purposes in mechanical design because, in many common situations, the dimensioning criteria involve local values (stresses, displacements, intensity factors, ...). Therefore, it is necessary to evaluate also the quality of these local quantities calculated by finite element analysis. Such an estimation of the local quality of a finite element numerical model remains a widely open investigation field. A first approach, proposed by Babuska and Strouboulis [4,5], is based on the concept of pollution error. Another approach is to use extraction operators [6–9], which depend on the type of local quantity considered. In general, these extraction operators are determined approximately using a finite element method. Another approach to the evaluation of the local quality of stresses was proposed in [6]. This method takes advantage of the characteristics of a new version of the estimators in constitutive relation for elasticity problems. An initial implementation for two-dimensional problems showed the advantage of this new version to be that it leads to local error estimates which, experimentally, are upperbounds of the actual local errors on stresses. Other 2D examples can be found in [10].

The object of the present paper is to propose an application of this approach to 3D elasticity problems meshed with 10-node tetrahedral element. In Section 2, we briefly recall the basics of error estimators in constitutive relation in elasticity. The local error measure in stress we propose is presented in Section 3. In Section 4, we outline the new version of the estimators in constitutive relation.

Several examples of applications for quadratic elements are presented in Section 5 along with a detailed analysis of the local effectivity indexes obtained. The results obtained for these elements confirm the usefulness of the approach already evidenced in previous works.

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2. Error in constitutive relation

2.1. Reference problem

Let us consider an elastic structure within a domain Ω bounded by $\partial\Omega$. The external actions on the structure are represented by

- a prescribed displacement $\underline{U}_{\mathrm{d}}$ on a subset $\partial_1\Omega$ of the boundary,
- a volume force density $\underline{f}_{\mathrm{d}}$ defined in Ω,
- a surface force density $\underline{F}_{\mathrm{d}}$ defined on: $\partial\Omega = \partial\Omega - \partial_1\Omega$.

We designate the material's Hooke's operator by $\mathbf{K}$. Thus, the problem can be formulated as follows: find a displacement field $\underline{U}$ and a stress field σ defined on Ω which verify

- the kinematic constraints:

$$\underline{U} \in \mathscr{U} \quad \underline{U}|_{\partial_1\Omega} = \underline{U}_{\mathrm{d}} \tag{1}$$

- the equilibrium equations:

$$\sigma \in \mathscr{S} \quad \forall \underline{U}^* \in \mathscr{U}_{\mathrm{ad},0} \tag{2}$$

$$\int_\Omega \mathrm{Tr}[\sigma\epsilon(\underline{U}^*)]\,\mathrm{d}\Omega = \int_\Omega \underline{f}_{\mathrm{d}}\underline{U}^*\,\mathrm{d}\Omega + \int_{\partial_2\Omega} \underline{F}_{\mathrm{d}}\underline{U}^*\,\mathrm{d}S \tag{3}$$

- the constitutive relation:

$$\sigma = \mathbf{K}\epsilon(\underline{U}) \tag{4}$$

$\mathscr{U}$ is the space in which the displacement field ($\mathscr{U} = [H'(\Omega)]^3$) is sought, $\mathscr{S}$ the space of the stress ($\mathscr{S} = [L^2(\Omega)]^6$), $\mathscr{U}_0$ the space of the fields in $\mathscr{U}$ which are zero on $\partial_1\Omega$ and where $\epsilon(\underline{U})$ represents the linearized deformation associated with the displacement

$$\epsilon(\underline{U}) = \tfrac{1}{2}(\mathrm{grad}\,\underline{U} + \mathrm{grad}\,\underline{U}^t)$$

The solution to the reference problem is designated by $(\underline{U}_{\mathrm{ex}}, \sigma_{\mathrm{ex}})$.

2.2. Definition of the error in constitutive relation

The approach based on the error in constitutive relation relies on a partitioning of the above equations into two groups

- the admissibility conditions: Eqs. (1) and (3),
- the constitutive relation: Eq. (4).

In practice, the constitutive relation is often the least reliable of all the equations of the reference problem. Therefore, it is natural to consider approximate solutions which verify the admissibility conditions exactly and to quantify quality by the extent to which the constitutive relations are verified. This leads us to the introduction of the following definition:

We say that fields $\hat{s} = (\widehat{U}, \hat{\sigma})$ are admissible if

- field $\widehat{U}$ verifies (1) ($\widehat{U}$ is kinematically admissible),
- field $\hat{\sigma}$ verifies (3) ($\hat{\sigma}$ is statically admissible).

One can then define a measure of the error in constitutive relation by

$$\hat{e}_h(\hat{s}) = \|\hat{\sigma} - \mathbf{K}\epsilon(\widehat{U})\|_\Omega \tag{5}$$

with $\|\bullet\|_\Omega = \int_\Omega \mathrm{Tr}[\bullet\mathbf{K}^{-1}\bullet]\,\mathrm{d}\Omega$ and a relative error by

$$\hat{\epsilon}_h = \frac{\|\hat{\sigma} - \mathbf{K}\epsilon(\widehat{U})\|_\Omega}{\|\hat{\sigma} + \mathbf{K}\epsilon(\widehat{U})\|_\Omega} \tag{6}$$

2.3. Errors in finite element discretization

Classically, if one discretizes the reference problem using a displacement-type finite element method, one obtains the following problem:

Find the kinematically admissible finite element displacement field $\underline{U}_h$ such that

$$\forall \underline{U}_h^* \in \mathscr{U}_{h0} \int_\Omega \mathrm{Tr}[\epsilon(\underline{U}_h)\mathbf{K}\epsilon(\underline{U}_h^*)]\,\mathrm{d}\Omega = \int_\Omega \underline{f}_{\mathrm{d}}\underline{U}_h^*\,\mathrm{d}\Omega + \int_{\partial_2\Omega} \underline{F}_{\mathrm{d}}\underline{U}_h^*\,\mathrm{d}S \tag{7}$$

where $\mathbf{U}_{h0}$ is the space of finite element displacement fields which are zero on $\partial_1\Omega$.

The corresponding stress field is

$$\sigma_h = \mathbf{K}\epsilon(\underline{U}_h) \tag{8}$$

The method to evaluate the errors due to the finite element discretization consists of reconstructing, starting from the finite element data and solution, an admissible pair $\hat{s} = (\underline{\widehat{U}}_h, \hat{\sigma}_h)$. Since the finite element field is kinematically admissible, one takes

$$\underline{\widehat{U}}_h = U_h \tag{9}$$

On the contrary, the stress field σ_h, is not statically admissible. Techniques to reconstruct admissible stress fields have been under development for several years [2,11]. They enable one to obtain a field $\hat{\sigma}_h$ which verifies the equilibrium equations exactly (3). We will discuss these techniques in the next section.

The error in constitutive relation associated with the admissible pair $\hat{s} = (\underline{\widehat{U}}_h, \hat{\sigma}_h)$ is

$$\hat{e}_h = \|\hat{\sigma}_h - \mathbf{K}\epsilon(\underline{U}_h)\|_\Omega = \|\hat{\sigma}_h - \sigma_h\|_\Omega \tag{10}$$

2.4. Property of the global error in constitutive relation

Using the Prager–Synge theorem [12], one shows easily that

$$\|\sigma_{\rm ex} - \sigma_h\|_\Omega \leqslant \hat{e}_h \tag{11}$$

Thus, the error in constitutive relation is an upperbound of the discretization error. In other words, the global effectivity index (ratio of the estimated error to the true error) is always greater than 1. If one introduces the stress field

$$\sigma_h^* = \tfrac{1}{2}(\hat{\sigma}_h + \sigma_h) \tag{12}$$

one also obtains,

$$\|\sigma_{\rm ex} - \sigma_h^*\|_\Omega = \tfrac{1}{2}\hat{e}_h \tag{13}$$

3. Estimation of the local errors on stresses

By construction, the square of the error in constitutive relation is the sum of contributions on each element of the mesh

$$\hat{e}_h^2 = \sum_{E\in\mathbf{E}} \hat{e}_{h,E}^2 \tag{14}$$

with

$$\hat{e}_{h,E} \equiv \|\hat{\sigma}_h - \sigma_h\|_E = \int_E \mathrm{Tr}[(\hat{\sigma}_h - \sigma_h)\mathbf{K}^{-1}(\hat{\sigma}_h - \sigma_h)]\,\mathrm{d}E$$

A priori, it was not possible to prove, on the local level, an inequality of the type (11)

$$\|\sigma_{\rm ex} - \sigma_h\|_E \leqslant \|\hat{\sigma}_h - \sigma_h\|_E \tag{15}$$

Nevertheless, in 2D, if one generates field $\hat{\sigma}_h$ using the method introduced in [13], and for linear tetrahedral element [10] one observes experimentally that

$$\|\sigma_{\rm ex} - \sigma_h\|_E \leqslant C\|\hat{\sigma}_h - \sigma_h\|_E \tag{16}$$

where C is a constant numerically close to 1.

One should point out that this type of local property can be completely lacking for ZZ2 estimators [6], in spite of the fact that these estimators have excellent global behavior. In the next section, we will extend to three-dimensional 10-nodes elements calculations the techniques developed in 2D in [13] for the construction of admissible fields. We will see in Section 5 that the estimator thus obtained has excellent local characteristics.

4. Construction of admissible fields in 3D

4.1. Principle of the construction

The technique to build stress fields $\hat{\sigma}_h$, which verify the equilibrium equation (3) exactly has become a classic [2,11]. It involves two stages:

The first stage consists of constructing on the sides of the elements surface force densities $\underline{\widehat{F}}_h$ to represent the stress vectors $\hat{\sigma}_h \underline{n}_E$ as

$$[\hat{\sigma}_h \underline{n}_E]|_\Gamma = \eta_E \underline{\widehat{F}}_h \text{ avec } \Gamma \in \partial E \tag{17}$$

where η_E is a function, constant on each side, whose value is either 1 or -1 and such that on the side common to two adjacent elements E and E': $\eta_E + \eta'_E = 0$.

Moreover, these force densities are generated in such a way that, on each element E in the mesh, the volume loads $\underline{f}_{\rm d}$ and the surface loads $\eta_E \underline{\widehat{F}}_h$ are in equilibrium on E. The detail of this construction procedure can be found in [11].

The second stage consists of constructing, element by element, a solution to the equilibrium equations

$$\begin{cases} \underline{\mathrm{div}}\,\hat{\sigma}_E + \underline{f}_{\rm d} = 0 & \text{in } E \\ \hat{\sigma}_E \underline{n}_E = \eta_E \widehat{F}_h & \text{on } \partial E \end{cases} \tag{18}$$

Then, field $\hat{\sigma}_h$ is obtained by: $\forall E \hat{\sigma}_h|_E = \hat{\sigma}_E$.

For a given set of densities, of all the solutions of (18) the best field $\hat{\sigma}_h|_E$ is the one which is solution to the minimization problem

$$\min_{\hat{\sigma}_E \text{ verifying (18)}} \|\hat{\sigma}_E - \sigma_h\|_E \tag{19}$$

By duality, this is equivalent to seeking a displacement field $\underline{V}_E$ defined on E such that

$$\underline{V}_E \in \mathscr{U}(E) \quad \text{and} \quad \forall \underline{V}^* \in \mathscr{U}(E)$$
$$\int_E \mathrm{Tr}[(\epsilon(\underline{V}_E)\mathbf{K}\epsilon(\underline{V}^*)]\,\mathrm{d}E = \int_E \underline{f}_{\rm d}\underline{V}^*\,\mathrm{d}E + \int_{\partial E} \eta_E \underline{\widehat{F}}_h \underline{V}^*\,\mathrm{d}\Gamma \tag{20}$$

where $\mathscr{U}(E)$ designates the space of restrictions to E of the fields in $\mathscr{U}$. Then $\hat{\sigma}_E$ is given by

$$\hat{\sigma}_E = \mathbf{K}\epsilon(\underline{V}_E) \tag{21}$$

Thus, one can obtain an approximation of $\hat{\sigma}_E$ by solving Problem (20) by a classical finite element method on E. In practice, it is sufficient, in order to obtain a good approximation, to consider either a discretization of E with a single element but an interpolation of degree $p+k$, where p is the degree of interpolation used in the finite element analysis and k a positive integer, or a subdivision of element E along with an interpolation of degree $p+k'$ (generally, $k<k'$).

The essential step is the construction of densities $\underline{\widehat{F}}_h$, as this conditions the quality of the resultant field $\hat{\sigma}_h$ and, consequently, the quality of the error estimator in constitutive relation.

5. Study of the quality of the local estimations

We present examples modeled with quadratic tetrahedral ($p=2$). In order to measure the local quality of our estimator, we have introduced a local effectivity index ζ_E

$$\zeta_E = \frac{\|\hat{\sigma}_h - \sigma_h\|_E}{\|\sigma_{\mathrm{ex}} - \sigma_h\|_E} \tag{22}$$

Here, we work on a simple test case where σ_{ex} is known analytically. This enables us to define ζ_E exactly and to measure the quality of our estimator.

5.1. Cube example

We consider an elastic cube fixed on face $\partial_1\Omega$ and subjected to second-degree polynomial forces on the five other faces $\partial_2\Omega$ (Fig. 1). There are no external body forces. The structure was meshed with forty TET10 elements (Fig. 2).

On this example, we found the exact solution. Indeed, the surface force densities were chosen such that the exact solution is

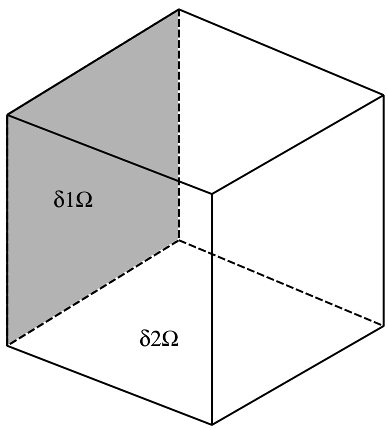

Fig. 1. Cube example.

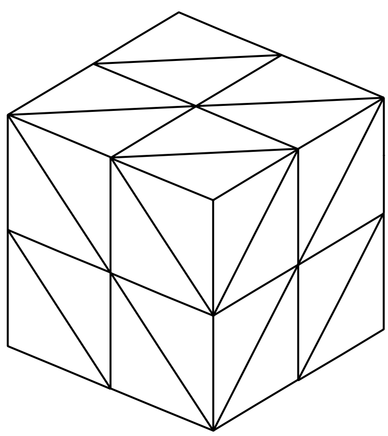

Fig. 2. Corresponding mesh.

$$\sigma_{\mathrm{ex}} = \begin{bmatrix} -2xy & \dfrac{y^2 - z^2}{2} - \dfrac{x^2}{1-v} & yz \\ \dfrac{y^2 - z^2}{2} - \dfrac{x^2}{1-v} & \dfrac{xy}{1-v} & 0 \\ yz & 0 & \dfrac{xy}{1-v} \end{bmatrix}_{(x,y,z)} \tag{23}$$

$$\underline{U}_{\mathrm{ex}} = \frac{1+v}{E(1-v)} \begin{bmatrix} -x^2 y \\ (1-v)x(y^2 - z^2) - \dfrac{x^3}{3} \\ 2(1-v)xyz \end{bmatrix}$$

The global error estimate for this structure was $\hat{\varepsilon}_h = 7\%$, whereas the exact error was $\epsilon = 2.7\%$, which corresponds to a global effectivity index $\zeta = 2.6$. Using the finite element solution and the exact solution, local indexes ζ_E can be calculated. For each element, one can plot the stresses $\|\sigma_h\|^2/\mathrm{mes}(E)$ (Fig. 3). The local error density (Fig. 4) and the local effectivity indexes ζ_E (Fig. 5); all these results are given element by element in order of increasing stresses.

For example,

Element #40 (the most highly loaded)	Element #18
Stress $\|\sigma_h\|^2/\mathrm{mes}(E) = 289$ MPa	Stress $\|\sigma_h\|^2/f\,\mathrm{mes}(E) = 24$ MPa
Estimated local error: 1%	Estimated local error: 14%
Exact local error: 0.7%	Exact local error: 8%

The results illustrate two characteristic properties of the estimator. On the one hand, in this example, all effectivity indexes are greater than one: this means that the local error estimation overestimates the actual error

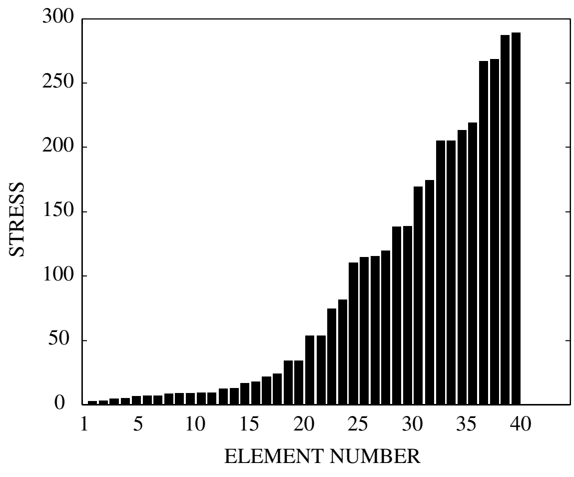

Fig. 3. Local stress density per element in order of increasing stresses.

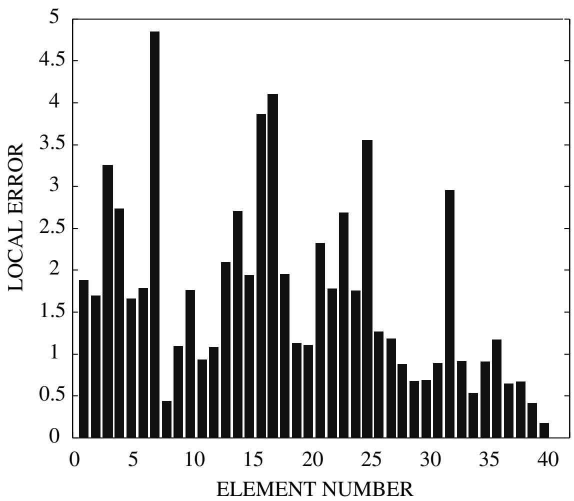

Fig. 4. Local error density in order of increasing stresses.

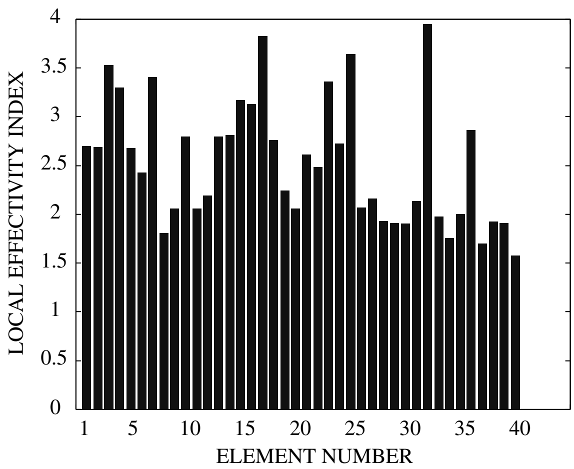

Fig. 5. Local effectivity indexes in elements in order of increasing stresses.

made in each element of the structure. On the other hand, all effectivity indexes are relatively close to 1: this means that the overestimation is not severe. It is also worth mentioning that the latter property is all the more true for elements whose stress (in the energy norm sense) is high. Besides, these results confirm the results obtained on examples meshed using four-node tetrahedral elements.

Fig. 6 shows the histograms of the effectivity indexes. The horizontal axis represents local effectivity index levels while bar heights represent the number of elements having the corresponding level. This graph shows clearly that a majority of elements have a local effectivity index close to 2.

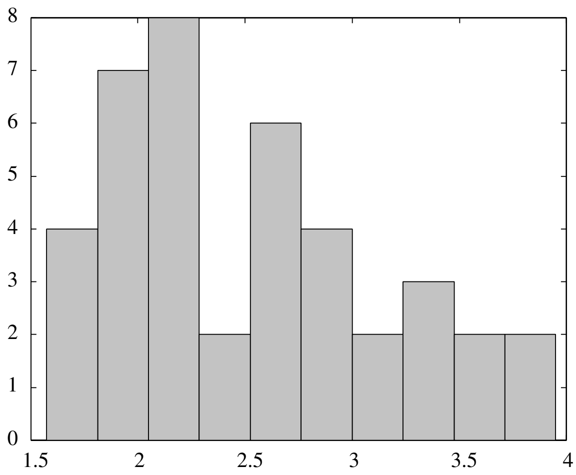

Fig. 6. Histogram.

5.2. Cylinder example

The second example is an elastic crown subjected to a set of internal and external pressures (Fig. 7). There are no external body forces. The structure was meshed with 648 10-node tetrahedral elements (Fig. 8). This is an axisymmetric problem; however, the whole structure was meshed and, therefore, did not benefit from the periodicity in θ. This enabled us to check that the estimation remained periodic, except for the inaccuracies of the mesh.

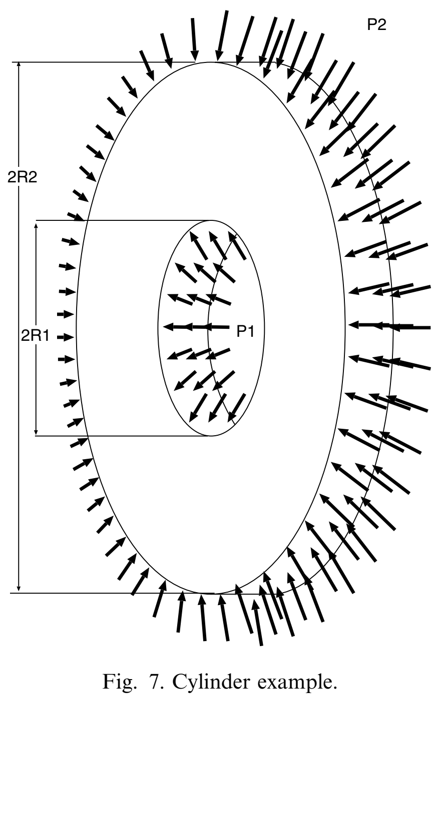

Fig. 7. Cylinder example.

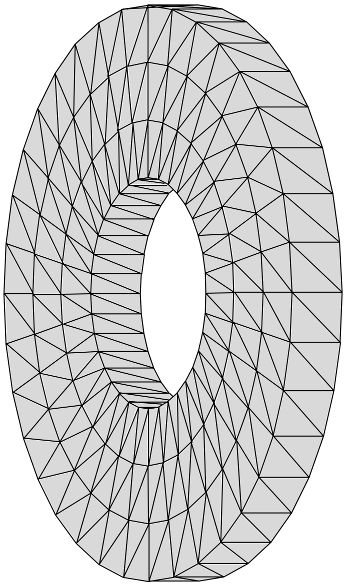

Fig. 8. Corresponding mesh.

The solution was expressed in cylindrical coordinates.

$$\sigma_{ex} = \begin{bmatrix} A - \frac{B}{r^2} & 0 & 0 \\ 0 & A + \frac{B}{r^2} & 0 \\ 0 & 0 & \frac{2\lambda}{\lambda+2\mu}A \end{bmatrix}_{(\underline{e_r},\underline{e_\theta},\underline{e_z})} \tag{24}$$

$$\underline{U}_{ex} = \begin{bmatrix} \frac{A}{\lambda+2\mu} r + \frac{B}{2\mu r} \\ 0 \\ 0 \end{bmatrix}_{(\underline{e_r},\underline{e_\theta},\underline{e_z})}$$

with

$$A = \frac{p_1 R_1^2 + p_2 R_2^2}{(R_2^2 - R_1^2)}$$

and

$$B = \frac{p_1 + p_2}{(R_1^{-2} - R_2^{-2})}$$

The numerical values used in this example were: $p_1 = 0$ MPa, $p_2 = 2.5$ MPa, $R_1 = 10$ mm and $R_2 = 25$ mm.

The global error estimate for this structure was $\epsilon_h = 9.5\%$, whereas the exact error was $\epsilon = 3.3\%$, which corresponds to a global effectivity index $\zeta = 2.8$. Using the finite element solution and the exact solution, local indexes ζ_E can be calculated. For each element, one can plot the local density error (Fig. 9) and the local effectivity indexes ζ_E (Fig. 10); all these results are given element by element in order of increasing stresses.

The results are given for a 10-degree section, which corresponds to the period of the mesh Fig. 10.

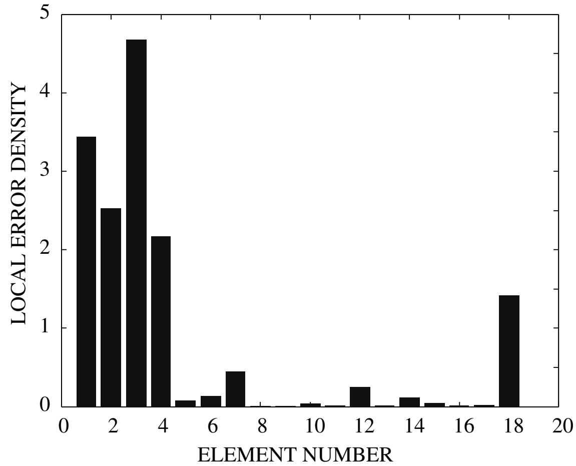

Fig. 9. Local error density in order of increasing stresses.

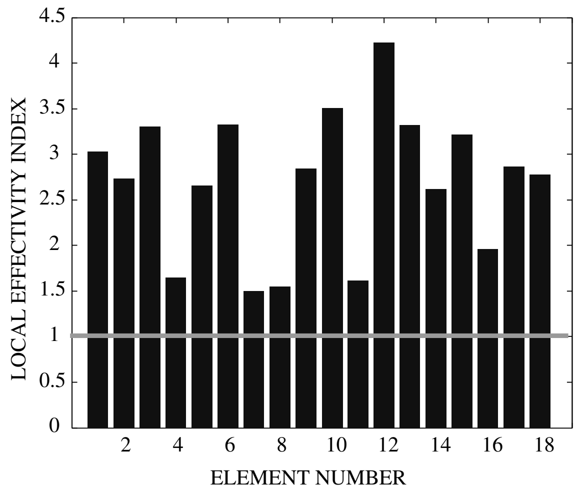

Fig. 10. Local effectivity indexes in the elements in order of increasing stresses.

For example,

Element #648 (the most highly loaded)	Element #110
Stress $\|\sigma_h\|^2/\mathrm{mes}(E) = 170$ MPa	Stress $\|\sigma_h\|^2/\mathrm{mes}(E) = 85$ MPa
Estimated local error: 8.4%	Estimated local error: 15.5%
Exact local error: 3.3%	Exact local error: 9.7%

In this example, the results illustrate again the two properties of the estimator mentioned earlier. On the one hand, all effectivity indexes are greater than one: this

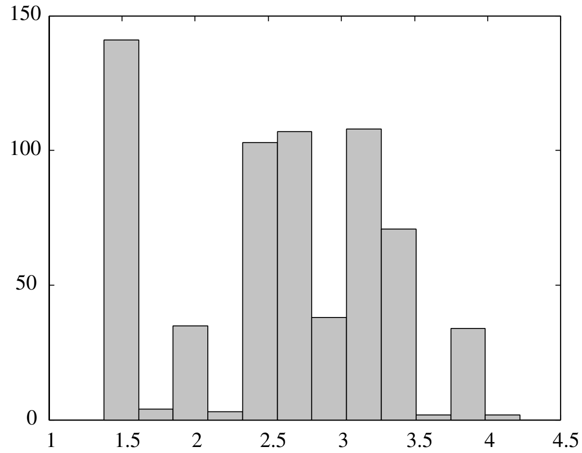

Fig. 11. Histogram.

means that for each element in the structure the local error estimation overestimates the actual error made. On the other hand, all effectivity indexes are relatively close to 1: this means that the over-estimation is not severe. It is also worth mentioning that the latter property is all the more true for elements whose stress (in the energy norm sense) is high.

In the same way as on the preceding example, one can observe on the histogram, that a majority of elements have a local effectivity index close to 1.5 (Fig. 11).

6. Conclusions

In this paper, we extended the results presented in [10] to the case of structures meshed with 10-node tetrahedral elements. We showed on the examples considered that the new generation of error in constitutive relation enables us to obtain locally a good estimation of the actual error on the stress. Indeed, the estimated error is close to the actual error. Moreover, we observed numerically on all the structures tested that this estimate is an upperbound of the actual error.

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