Adaptive meshing for local quality of FE stresses

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Abstract

Purpose – In this paper, we focus on the quality of a 2D elastic finite element analysis.

Design/methodology/approach – Our objective is to control the discretization parameters in order to achieve a prescribed local quality level over a dimensioning zone. The method is based on the concept of constitutive relation error.

Findings – The method is illustrated through 2D test examples and shows clearly that in terms of cost, this technique provides an additional benefit compared to previous methods.

Research limitations/implications – The saving would be even more significant if this mesh adaptation technique were applied in three dimensions. Indeed, in 3D problems, the computing cost is vital and, in general, it is this cost that sets the limits.

Practical implications – This tool is directly usable in the design stage.

Originality/value – The new tool developed guarantees a local quality level prescribed by the user.

Keywords Finite element analysis, Error analysis, Meshes

Paper type Research paper

1. Introduction

The primary objective in developing discretization error estimators is to control the quality of a finite element analysis. However, the information thus obtained (evaluation of the global error, contribution of each element to the global error) enables one to develop procedures for adapting the calculation parameters in order to achieve the desired overall level of accuracy while reducing calculation costs. Naturally, such procedures have been developed mainly for linear analysis and, in this context, they have led to very robust mesh adaptation techniques (Ladevèze *et al.*, 1991; Coorevits *et al.*, 1996). Parameter adaptation techniques for nonlinear and dynamic problems can be found in Ladevèze and Oden (1998), Aubry *et al.* (1999) and Gallimard and Pelle (2002). In recent years, estimators allowing the control of the quality of local quantities (stresses, displacements, contour integrals) have been proposed by different teams (Rannacher and Suttmeier, 1997; Peraire and Patera, 1998; Prudhomme and Oden, 1999; Cirak and Ramm, 1998; Ladevèze *et al.*, 1999; Strouboulis *et al.*, 2000; Stein *et al.*, 2001).

Naturally, parameter adaptation techniques have been associated with these estimators (Rannacher and Suttmeier, 1997; Prudhomme and Oden, 1999; Stein *et al.*, 2001). The objective of this paper is to propose a mesh adaptation technique for linear problems which enables one to guarantee the quality of the stresses calculated in a dimensioning zone while reducing the cost of the finite element analyses. This technique is based on the local estimator developed at LMT-Cachan (Ladevèze and Rougeot, 1997; Ladevèze *et al.*, 1999; Florentin *et al.*, 2002). In the first part of this paper, after having reviewed the classical formulation of a linear elasticity problem, we outline the basic principles of the error in constitutive relation and its application to the estimation of the local quality of the calculated finite element stresses. In the second part, we review the mesh adaptation technique which enables one to control the global quality of a finite element analysis. In the third part, we present the modifications which we made to this technique in order to guarantee a given quality level in a zone specified by the user (*a priori* a dimensioning zone) while reducing the cost of the finite element analysis. First examples of applications to two-dimensional problems are presented. They clearly show the interest and the effectiveness of the proposed method.

2. Error in constitutive relation for linear problems

2.1 The reference problem

Let us consider an elastic structure occupying a domain Ω bounded by $\partial\Omega$. The actions of the environment on the structure are represented by:

- a prescribed displacement $\underline{U}_d$ over a part $\partial_1\Omega$ of the boundary;
- a prescribed volume force density $\underline{f}_d$ in Ω; and
- a prescribed surface force density $\underline{F}_d$ over $\partial_2\Omega = \partial\Omega - \partial_1\Omega$.

The Hooke's operator of the material is denoted by $\mathbf{K}$. Then, the problem can be formulated as follows:

Find a displacement field $\underline{U}_{ex}$ and a stress field $\boldsymbol{\sigma}_{ex}$ defined over Ω which verify:

- *the kinematic constraints:*

$$\underline{U}_{ex}|_{\partial_1\Omega} = \underline{U}_d \tag{1}$$

- *the equilibrium equations:* for any $\underline{U}^*$ zero over $\partial_1\Omega$

$$\int_\Omega \mathrm{Tr}[\boldsymbol{\sigma}_{ex}\boldsymbol{\varepsilon}(\underline{U}^*)]\,\mathrm{d}\Omega = \int_\Omega \underline{f}_d\underline{U}^*\,\mathrm{d}\Omega + \int_{\partial_2\Omega} \underline{F}_d\underline{U}^*\,\mathrm{d}S \tag{2}$$

- *the constitutive relation:*

$$\boldsymbol{\sigma}_{ex} = \mathbf{K}\boldsymbol{\varepsilon}(\underline{U}_{ex}) \tag{3}$$

where $\boldsymbol{\varepsilon}(\underline{U})$ represents the linearized strain associated with the displacement. $(\underline{U}_{ex},\boldsymbol{\sigma}_{ex})$ is the solution pair of this reference problem. $(\underline{U}_h, \boldsymbol{\sigma}_h)$ is the approximate finite element solution pair of this reference problem.

2.2 Discretization error

We want to know the quality of the discretized solution $(\underline{U}_h, \boldsymbol{\sigma}_h)$ as an approximation of the solution $(\underline{U}_{ex}, \boldsymbol{\sigma}_{ex})$ of the corresponding continuous problem. Classically, one defines a measure of the solution error over the structure as:

$$e_h = \|\underline{e}_h\|_{u,\Omega} = \|\underline{U}_{ex} - \underline{U}_h\|_{u,\Omega} = \|\boldsymbol{\sigma}_{ex} - \boldsymbol{\sigma}_h\|_{\sigma,\Omega} \tag{4}$$

where $\|\bullet\|_{\bullet,\Omega}$ represents the energy norm of $\bullet$ over Ω.

e_h provides only some global scalar energy information on the quality of the finite element calculation. It could be interesting to decompose e_h taking into account the finite element partition introduced, and to break down the error into contributions from each element E of the mesh τ_h:

$$e_h^2 = \sum_{E \in \tau_h} e_{h,E}^2 \qquad (5)$$

with:

$$e_{h,E}^2 = \|\underline{e}_h\|_{u,E} = \|\underline{U}_{ex} - \underline{U}_h\|_{u,E} = \|\sigma_{ex} - \sigma_h\|_{\sigma,E} \qquad (6)$$

Remark. $e_{h,E}$ is a local error measure over element E: $e_{h,E} = 0 \Leftrightarrow \sigma_h = \sigma_{ex}$ in E.

2.3 Error in constitutive relation

The discretization error is estimated through the error in constitutive relation. The concept of error in constitutive relation relies on splitting the equations into two groups:

- the admissibility equations: constraints and equilibrium (1, 2); and
- the constitutive relation (3).

The constitutive relation has a particular status. In practice, this is often the least reliable equation. An admissible pair $(\hat{\underline{U}}, \hat{\sigma})$ verifying the first group of equations is constructed. Then, the non-verification of the constitutive relations enables one to define e_{CR}. This estimate is a sum of elementary contributions:

$$e_{\mathrm{CR}}^2 = \sum_{E \in \mathbf{E}} e_{\mathrm{CR},E}^2$$

$$e_{\mathrm{CR},E}^2 = \int_E \mathrm{Tr}[(\hat{\sigma} - \mathbf{K}\varepsilon(\hat{\underline{U}}))\mathbf{K}^{-1}(\hat{\sigma} - \mathbf{K}\varepsilon(\hat{\underline{U}}))]\,\mathrm{d}E = \|\hat{\sigma} - \mathbf{K}\varepsilon(\hat{\underline{U}})\|_{\sigma,E}^2 \qquad (7)$$

2.4 Implementation of the error in constitutive relation

2.4.1 The finite element displacement method. Since the pair $(\underline{U}_h, \sigma_h)$ is not *admissible*, in order to use an error in constitutive relation it is necessary to construct an *admissible* pair $(\underline{U}_h, \sigma_h)$ starting from the finite element solution and the problem's data.

- In the framework of a displacement finite element method of the conforming type, the displacement field $\underline{U}_h$ is admissible. For the sake of simplicity, one chooses:

$$\hat{\underline{U}} = \underline{U}_h \quad \text{in } \Omega \qquad (8)$$

- However, the stress field σ_h is not admissible (i.e. it does not verify the equilibrium exactly). The techniques used for the construction of an equilibrated stress field σ_h starting from $\hat{\sigma}$ and the problem's data are detailed in Ladevèze *et al.* (1991), Ladevèze and Rougeot (1997) and Florentin *et al.* (2002).

2.5 Relationship with the solution error

With this choice of admissible fields, the error in constitutive relation can be connected to the solution error through Prager-Synge's (1947) hypercircle theorem:

$$e_{\mathrm{CR}}^2 = \|\hat{\sigma} - \sigma_h)\|_{\sigma,\Omega}^2 = \|\hat{\sigma} - \sigma_{ex})\|_{\sigma,\Omega}^2 + \|\sigma_{ex} - \sigma_h)\|_{\sigma,\Omega}^2 \tag{9}$$

In particular, this theorem yields the following inequalities:

$$e_h \leq e_{\mathrm{CR}} \tag{10}$$

and

$$\|\sigma_{ex} - \sigma_h)\|_{\sigma,\Omega} \leq e_{\mathrm{CR}} \tag{11}$$

2.6 Local quality of the finite element solution

In Florentin *et al.* (2002, 2003), the quantity $e_{\mathrm{CR},E}$ is shown to be an upper bound of the error actually made $e_{h,E}$. The idea is that if one builds a better quality field $\hat{\sigma}$ than σ_h, i.e. such that:

$$\|\hat{\sigma} - \sigma_{ex}\|_{\sigma,E} = A\|\sigma_h - \sigma_{ex}\|_{\sigma,E} \quad \text{with } 0 \leq A < 1 \tag{12}$$

then, one obtains:

$$\|\sigma_h - \sigma_{ex}\|_{\sigma,E} \leq \|\sigma_h - \hat{\sigma}\|_{\sigma,E} + A\|\sigma_h - \sigma_{ex}\|_{\sigma,E} \tag{13}$$

Therefore, there exists a constant C such that:

$$\|\sigma_h - \sigma_{ex}\|_{\sigma,E} \leq C\|\hat{\sigma} - \sigma_h\|_{\sigma,E} \quad \text{with } C = \frac{1}{1-A} \tag{14}$$

Since $A < 1$, the upper bound (14) is obtained with $C \geq 1$. Furthermore, the smaller A is, the closer C is to 1. The construction of $\hat{\sigma}$, which in practice leads to equation (14), is detailed in Ladevèze and Rougeot (1997) and Florentin *et al.* (2002, 2003). The stress $\hat{\sigma}$ is obtained by an optimisation procedure over a family of stresses in equilibrium with external forces. In Ladevèze and Rougeot (1997) and Florentin *et al.* (2002, 2003) numerical tests on numerous test cases show that in practice equation (12) is verified.

3. Adapted mesh for controlling the quality of the finite element stresses

In this section, we present the methods which enable one to choose the discretization parameters in order to achieve a given quality level. This quality can be global (but, in this case, the practical benefits are limited) or local, in which case the discretization obtained leads to a result of direct value to the user.

3.1 Adapted mesh for controlling the global quality of a linear calculation

The idea consists in utilizing the results of an initial finite element analysis and the associated error estimates to determine an optimum mesh, i.e. a mesh which provides the desired overall accuracy while minimizing the computing costs.

In short, the principle of the procedure is the following:

- one performs an initial calculation on a relatively coarse mesh τ;
- on this mesh τ, one calculates the relative global error ε as well as the contributions ε_E; and
- one uses this information to determine the characteristics of the optimum mesh $\tau*$.

3.1.1 The optimum mesh. The optimum mesh τ^*, whose concept was introduced in Ladevèze (1977) and Ladevèze and Leguillon (1981), is such that:

$$\varepsilon^* = \varepsilon_d \text{ (prescribed accuracy)} \quad \varepsilon_E^* \text{ uniform over } \tau^* \tag{15}$$

This definition is equivalent to having an evenly distributed error. This criterion does not necessarily correspond to minimum computing costs. A criterion which does not have this drawback, introduced in Ladevèze *et al.* (1986), consists in taking the following definition:

$$\varepsilon^* = \varepsilon_0 \text{ (prescribed accuracy)} \quad N^*\text{, the number of elements of } \tau^*\text{, is minimum} \tag{16}$$

This is the criterion that we choose, since it leads naturally to minimum computing costs.

3.1.2 Determination of the optimum mesh. The idea, in order to determine the characteristics of the optimum mesh τ^*, is to calculate for each element E of the mesh τ a size modification coefficient:

$$r_E = \frac{h_E^*}{h_E} \tag{17}$$

where h_E is the current size of element E and h_E^* the (unknown) size the elements of τ^* must have in the zone corresponding to element E in order to achieve optimality (Figure 1).

Thus, the determination of the optimum mesh is equivalent to the determination of the initial mesh of a map of size modification coefficients.

The calculation of the coefficients r_E is based on the rate of convergence of the error:

$$\varepsilon = O(h^q) \tag{18}$$

where q depends on the type of element being used, but also on the regularity of the exact solution of the problem being considered. In this case, in order to predict the maps of optimum sizes, one writes that the ratio of the contributions to the error is:

$$r_E^q = \left(\frac{h_E^*}{h_E}\right)^q = \frac{\varepsilon_E^*}{\varepsilon_E} \tag{19}$$

where ε_E^* is the contribution of the elements of τ^* located in the zone of E:

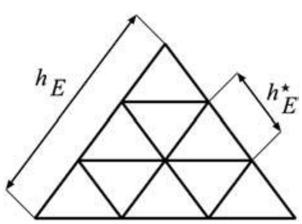

Figure 1.
Element sizes

$$\varepsilon_E^* = \left[\sum_{E^* \subset E} \varepsilon_{E^*}^2 \right]^{\frac{1}{2}} \tag{20}$$

The square of the global relative error on mesh τ can be evaluated by:

$$\varepsilon^{*2} = \sum_E \varepsilon_E^{*2} = \sum_E r_E^{2q} \varepsilon_E^2 \tag{21}$$

and the number of elements of mesh τ^* by:

$$N^* = \sum_E \frac{1}{r_E^d} \tag{22}$$

where d is the dimension of the space. (In the 2D case being considered, $d = 2$.)

Thus, the problem to be solved is:

$$\text{Minimize } (N^*) \text{ with } \sum_E r_E^{2q} \varepsilon_E^2 = \varepsilon_d^2 \tag{23}$$

One can show (Ladevèze and Pelle, 2004) that the resolution of such a problem yields the size map $\{r_E\}_{E \varepsilon \tau}$, with:

$$r_E = \frac{\varepsilon_d^{\frac{1}{q}}}{\varepsilon_E^{\frac{2}{2q+d}} \left[\sum_E \varepsilon_E^{\frac{2d}{2q+d}} \right]^{\frac{1}{2q}}} \tag{24}$$

3.1.3 Optimality check. Having determined the optimum sizes and constructed the optimum mesh τ^*, one performs a new finite element calculation. In order to assess the effectiveness of the different procedures used, it is necessary to check the optimality of mesh τ^*.

A first verification consists simply in calculating the discretization error ε^* of mesh τ^*. If this error is not close to the desired accuracy, the constructed mesh is certainly not optimum.

A simple technique to check optimality consists in determining a new size map for a prescribed accuracy equal to the accuracy ε^* actually obtained. For an accuracy ε^*, if the mesh is strictly optimum, this procedure must yield size modification coefficients such that:

$$\forall E^* \in \tau^*, \quad r_E^* = 1 \tag{25}$$

In actual situations, such mesh quality is never achieved. Indeed:

- the size prediction assumes that the elements are regular (equilateral triangles), which is not true in practice; and
- the mesh generator does not always achieve the prescribed size map exactly.

These mesh adaptation methods with global target error can be found in Ladevèze *et al.* (1991) and Coorevits *et al.* (1994, 1996).

3.2 Adapted mesh for controlling the local quality of a linear calculation

The mesh obtained by adaptation guarantees a given global error over the structure. This enables one to obtain a better-quality finite element analysis by setting a lower target error. Here, we propose an approach which still aims to achieve a global error level, but, in addition, also guarantees a given quality level in a given zone ω. Thus, the user can obtain an optimized mesh which provides, at minimum cost, local information with a given accuracy. This is done by first setting the element sizes in zone ω in order to achieve the desired local quality, then minimizing the number of elements in the whole structure.

The principle of the procedure is the following:

- one performs an initial calculation on a relatively coarse mesh τ;
- on this mesh τ, one calculates the relative global error ε as well as the relative local error $e_{\text{loc}(\omega)}$ over zone ω; and
- one uses this information to determine the characteristics of the local-optimum mesh τ^*_{loc}.

The local relative error $e_{\text{loc}(\omega)}$ over ω is defined by the ratio:

$$e_{\text{loc}(\omega)} = \frac{\|\hat{\sigma} - \sigma_h\|_{\sigma,\omega}}{\|\hat{\sigma} + \sigma_h\|_{\sigma,\omega}} \tag{26}$$

In particular, over a zone corresponding to an element E,

$$e_{\text{loc}(E)} = \frac{\|\hat{\sigma} - \sigma_h\|_{\sigma,E}}{\|\hat{\sigma} + \sigma_h\|_{\sigma,E}} \tag{27}$$

Remark. The local relative error which one would want to define is:

$$\widetilde{e_{\text{loc}(E)}} = \frac{\|\sigma_{ex} - \sigma_h\|_{\sigma,E}}{\|\sigma_{ex} + \sigma_h\|_{\sigma,E}} \tag{28}$$

but, since σ_{ex} is not known, only $e_{\text{loc}(E)}$ can be defined. In practice, $e_{\text{loc}(E)}$ is a good estimate of $\widetilde{e_{\text{loc}(E)}}$.

3.2.1 Local-optimum mesh. A local-optimum mesh τ^*_{loc} is such that:

$$\begin{aligned} &\varepsilon^* = \varepsilon_d \text{ (prescribed global accuracy)} \quad e^*_{\text{loc}(\omega)} = e_{\text{loc},d} \text{ (prescribed local accuracy)} \\ &N^*, \text{ the number of elements of } \tau^*_{\text{loc}}, \text{ is minimum} \quad e^*_{\text{loc}(E)} \text{ is uniform over } \omega \end{aligned} \tag{29}$$

This criterion leads naturally to minimum cost while guaranteeing good-quality information for all elements in zone ω. In a mesh thus optimized, each element E^* within zone ω has a local error equal to $e_{\text{loc},d}$.

One should note that:

$$e^*_{\text{loc}(E)} \text{constant} \quad \forall E \in \omega \Rightarrow e^*_{\text{loc}(E)} = e^*_{\text{loc}(\omega)} = e_{\text{loc},d} \quad \forall E \in \omega \tag{30}$$

This means that each local error has the prescribed value $e_{\mathrm{loc},d}$ over each element of zone ω.

$e_{\mathrm{loc},d}$ is fixed by the user. ε_d is a parameter to determine by numerical experiments. The first tests seem to show that choosing ε_d two or three times larger than $e_{\mathrm{loc},d}$ leads to good results.

3.2.2 Determination of the local-optimum mesh. Thus, the determination of the local-optimum mesh τ^*_{loc} is equivalent to the determination over the initial mesh of a size modification map with size r_E. In fact, this is a simple modification of the global case. Indeed, the optimization is performed in two steps:

(1) one sets the sizes in ω; and

(2) one minimizes the number of elements over $\Omega - \omega$.

The calculation of the coefficients r_E is based on the convergence rate of the error:

$$\varepsilon = \mathrm{O}(h^q) \tag{31}$$

where q depends on the type of element being used, but also on the regularity of the exact solution of the problem. In this case, in order to predict the optimum size maps, one writes:

$$r_E^q = \left(\frac{h_E^*}{h_E}\right)^q = \frac{\varepsilon_E^*}{\varepsilon_E} \tag{32}$$

where ε_E are the global relative contributions over an element E of mesh τ and ε_E^* are the corresponding quantities defined over τ^*: ε_E^* is the contribution to the global relative error of the elements of τ^* located in zone E:

$$\varepsilon_E^* = \left[\sum_{E^*\subset E} \varepsilon_{E^*}^2\right]^{\frac{1}{2}} \tag{33}$$

In addition:

$$\varepsilon_E^* = \frac{\|\hat{\sigma}^* - \sigma_h^*\|_{\sigma,E}}{\|\hat{\sigma}^* + \sigma_h^*\|_{\sigma,\Omega}} = \underbrace{\frac{\|\hat{\sigma}^* - \sigma_h^*\|_{\sigma,E}}{\|\hat{\sigma}^* + \sigma_h^*\|_{\sigma,E}}}_{e^*_{\mathrm{loc}(E)}=e_{\mathrm{loc},d}} \underbrace{\frac{\|\hat{\sigma}^* + \sigma_h^*\|_{\sigma,E}}{\|\hat{\sigma}^* + \sigma_h^*\|_{\sigma,\Omega}}}_{\Psi} \tag{34}$$

The term Ψ is evaluated by assuming that the energy does not change significantly from one mesh to the other:

$$\Psi = \frac{\|\hat{\sigma}^* + \sigma_h^*\|_{\sigma,E}}{\|\hat{\sigma}^* + \sigma_h^*\|_{\sigma,\Omega}} \approx \frac{\|\hat{\sigma} + \sigma_h\|_{\sigma,E}}{\|\hat{\sigma} + \sigma_h\|_{\sigma,\Omega}} \tag{35}$$

Using equalities (32) and (34), one obtains the size modification coefficients for the elements E located on ω:

$$r_E^q = \frac{e_{\mathrm{loc},d}\Psi}{\varepsilon_E} \approx \frac{e_{\mathrm{loc},d}}{e_{\mathrm{loc}(E)}} \tag{36}$$

i.e. finally:

$$r_E = \left(\frac{e_{loc,d}}{e_{loc(E)}}\right)^{\frac{1}{q}} \quad \forall E \in \omega \tag{37}$$

Then, in order to determine the r_E in zone $\Omega - \omega$, one proceeds in the same way as for the determination of the global-optimum mesh.

Thus, the problem to be solved is:

$$\text{Minimize } (N^*) \text{ with } \begin{cases} \sum_E r_E^{2q} \varepsilon_E^2 = \varepsilon_d^2 \\ r_E = \left(\frac{e_{loc,d}}{e_{loc(E)}}\right)^{\frac{1}{q}} \quad \forall E \in \omega \end{cases} \tag{38}$$

Introducing the known quantity $\tilde{\varepsilon}_d^2$:

$$\tilde{\varepsilon}_d^2 = \varepsilon_d^2 - \sum_{E \in \omega} r_E^{2q} \varepsilon_E^2 \tag{39}$$

the problem to be solved remains of the same type as for the global-optimum mesh:

$$\text{Minimize } (N^*) \text{ with } \sum_{E \in \Omega - \omega} r_E^{2q} \varepsilon_E^2 = \tilde{\varepsilon}_d^2 \tag{40}$$

Thus, one gets the size map for the elements located outside zone ω.

$$r_E = \frac{\tilde{\varepsilon}_d^{\frac{1}{q}}}{\varepsilon_E^{\frac{2}{2q+d}} \left[\sum_E \varepsilon_E^{\frac{2d}{2q+d}}\right]^{\frac{1}{2q}}} \quad \forall E \in \Omega - \omega \tag{41}$$

3.2.3 Optimality check. Having determined the optimum sizes and constructed the optimum mesh τ_{loc}^*, one performs a new finite element calculation. In order to assess the effectiveness of the different procedures carried out, one must check the optimality of mesh τ_{loc}^*.

The first verification consists calculating the discretization errors ε^* and $e_{loc(\omega)}^*$ of mesh τ_{loc}^*. If these errors are not close to the desired accuracies ε_d on the global level and $e_{loc(\omega),d}$ on the local level, the constructed mesh is certainly not optimum.

A simple technique in order to check optimality consists in determining a new size map for a prescribed accuracy equal to the accuracy actually obtained ε^* and $e_{loc(\omega)}^*$. If the mesh is exactly optimum, this procedure must yield size modification coefficients such that:

$$\forall E^* \in \tau_{loc}^*, \quad r_E^* = 1 \tag{42}$$

3.3 Examples of implementation

3.3.1 Beam in bending. This example illustrates the method on a beam subjected to bending.

The part is dimensioned using the yield stress in the most highly solicited zone. One is seeking an optimum mesh in order to obtain good-quality stresses in the zone where the mechanical loading is maximum. The geometry and the loading are shown in Figure 2.

A coarse and regular initial mesh is constructed in order to get an idea of the most highly solicited zones at low cost. The zone with the highest stresses, as expected, is the central zone. We are more particularly interested in the compression zone (Figure 2).

For this initial mesh, the local (for the zone of interest) and global quality levels are given in Table I.

The first step consists seeking the mesh which yields 2 percent global error with the minimum number of elements. The corresponding mesh is shown in Figure 3.

The local and global quality levels obtained with this mesh are given in Table II.

Now, let us consider a mesh with the same local quality level over the zone of interest, i.e. 1 percent. We seek the mesh which guarantees this local quality level for all elements in the zone and, at the same time, yields 5 percent global error while, of course, minimizing the number of elements used. The corresponding mesh is shown in Figure 4.

The local and global quality levels obtained with this mesh are given in Table III.

In order to obtain this mesh, we performed an optimality check. Figure 5 shows the optimality histogram obtained on the local optimum mesh. On this graph, the number

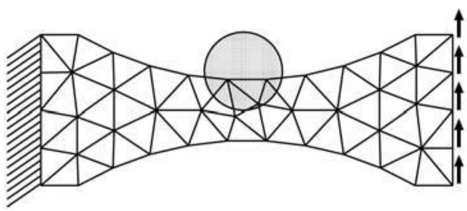

Figure 2.
The initial mesh

Table I.
The corresponding quality levels

Global error	6.60 percent
Number of elements	60
Number of nodes	151
Local error	3.99 percent

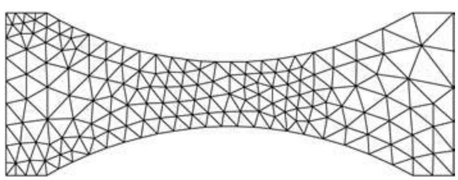

Figure 3.
The global-optimum mesh

Table II.
Global-optimum mesh: quality

Global error	1.58 percent (2 percent prescribed)
Number of elements	299
Number of nodes	666
Local error	0.9 percent (for information)

of elements of τ^* with a given r_E^* is plotted against r_E^*. One can see that r_E^* is sufficiently close to 1.

3.3.2 Load cell. This example concerns a load cell. The loading applied in order to study the deformation of the structure consists in a fixed condition in the lower right and a uniform pressure applied on the upper left (Figure 6).

One can observe that one zone follows a quasi-translation movement with respect to the other (Figure 6). In fact, four elastic pin joints forming a trapezoid enable this displacement to occur. For the purpose of dimensioning, we are interested in the most highly solicited joint alone (zone of interest: lower right).

For this loading and the initial mesh, one obtains the initial quality level shown in Figure 7.

Now, let us consider the global-optimum mesh (Figure 8). This mesh, of course, is more refined in the four elastic joints and less refined elsewhere.

If one changes the objective and seeks to guarantee a local quality level as well, one obtains the mesh of Figure 9.

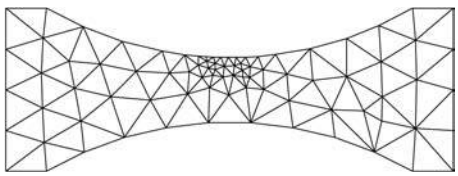

Figure 4. The local-optimum mesh

Global error	4.61 percent (5 percent prescribed)
Number of elements	130
Number of nodes	299
Local error	0.8 percent (1 percent prescribed)

Table III. Local-optimum mesh: quality

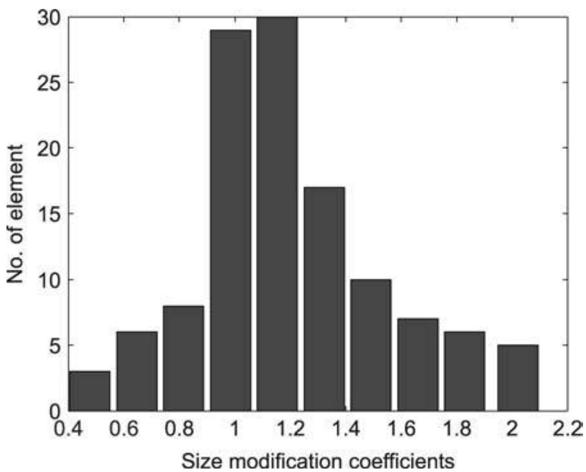

Figure 5. Optimality check for the local-optimum mesh

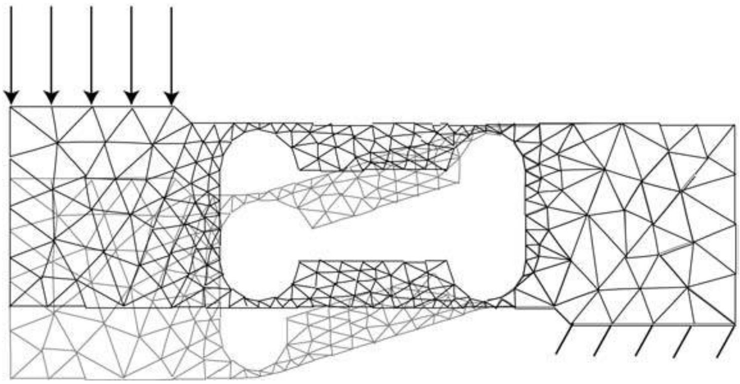

Figure 6.
Deformed shape

Global error	40%
No. of elements	337
Local error	50%

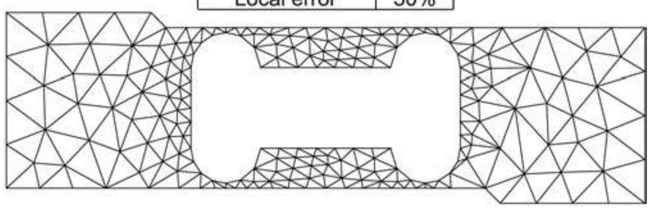

Figure 7.
Load cell: initial mesh

Global error	2.2% (2% prescribed)
No. of elements	1628
Local error	2.6% (for information)

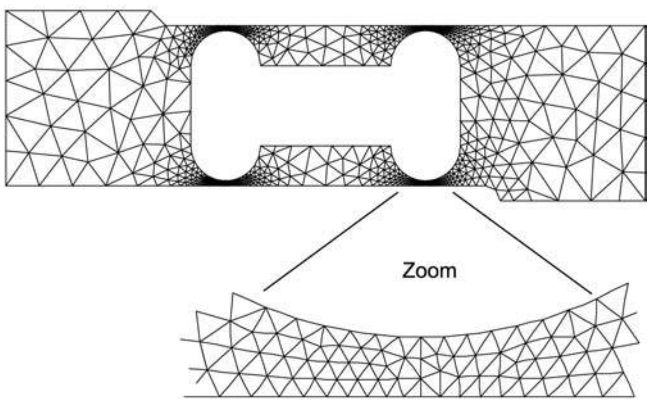

Figure 8.
Load cell: global-optimum mesh

3.3.3 Elastic connection. The elastic connection being studied is part of a test assembly (Figure 10) designed by Sébastien Le Loch (*LMT-Cachan*) for a series of experiments in cooperation with *EADS-LV* and *CNES*. The test piece is loaded in four-point bending, and the dimensioning of the elastic joints was carried out using a simple yield stress

Global error	9.13% (10% prescribed)
No. of elements	702
Local error	1.82% (2% prescribed)

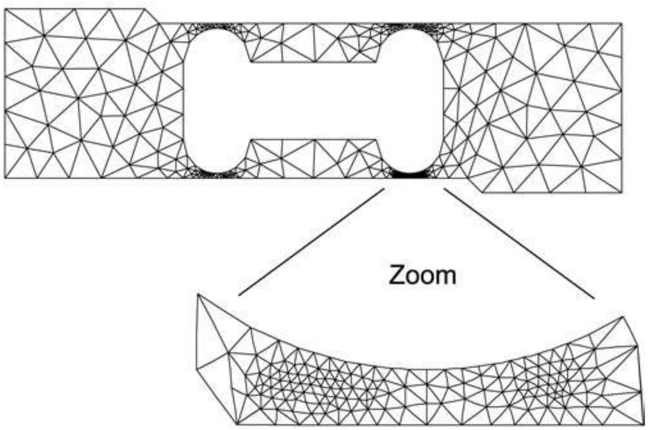

Figure 9.
Load cell: local-optimum mesh

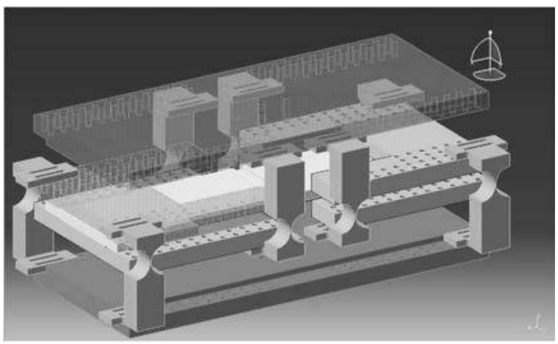

Figure 10.
Test assembly

criterion. Again, a local-optimum mesh obtained through the technique described in this paper is compared to the global-optimum mesh.

The boundary conditions are shown in Figure 11.

A coarse mesh is defined, which enables one to find the zone (or zones) with the highest stresses. The most highly loaded zone is, of course, the center zone with the

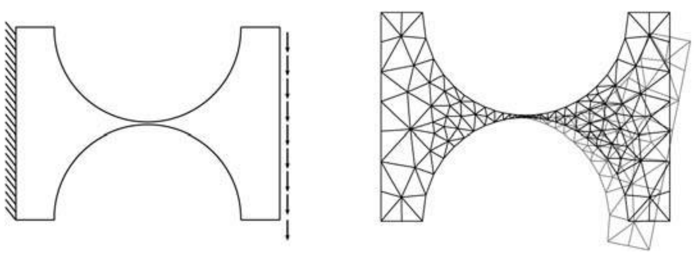

Figure 11.
Boundary conditions and deformed shape

smallest cross-section. The different quality levels obtained on this mesh are shown in Figure 12.

The quality level obtained for the global-optimum mesh is shown in Figure 13.

The quality level obtained for the local-optimum mesh is shown in Figure 14.

4. Conclusions

The test cases which were carried out show several things. First of all, we have a new tool which guarantees a local quality level prescribed by the user. This tool is directly usable in the design stage. Then, in terms of cost, this technique provides an additional benefit compared to previous methods. Indeed, remeshing methods based on a global objective, by creating an adapted mesh, result in considerable time savings; remeshing methods based on a local objective, as we saw, result in a saving in terms of number of elements. This time saving is significant, particularly in the design phase, where it allows for several constructive solutions to be tested. The saving would be even more

Global error	21%
No. of elements	166
Local error	50% (for information)

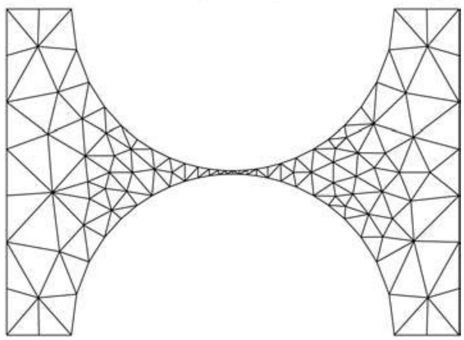

Figure 12.
Connection: initial mesh

Global error	2.1% (2% prescribed)
No. of elements	354
Local error	1.39% (for information)

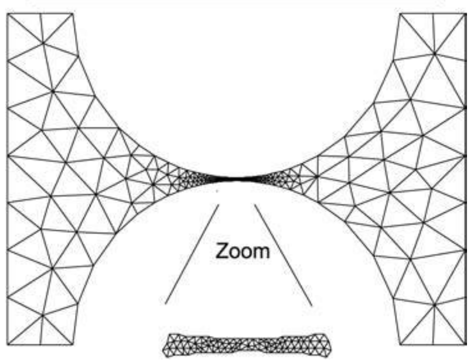

Figure 13.
Connection:
global-optimum mesh

Global error	4.88% (5% prescribed)
No. of elements	196
Local error	1.82% (2% prescribed)

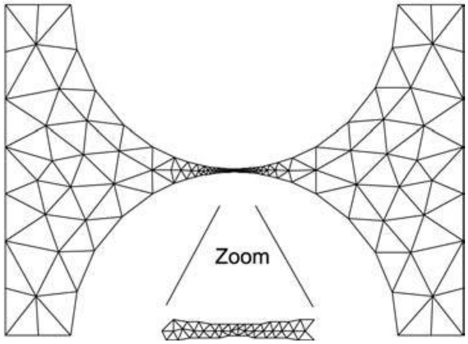

Figure 14.
Connection: local-optimum mesh

significant if this mesh adaptation technique were applied in three dimensions. Indeed, in 3D problems, the computing cost is vital and, in general, it is this cost that sets the limits.

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