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Research article

Improved results on an extended dissipative analysis of neural networks with additive time-varying delays using auxiliary function-based integral inequalities

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Abstract: The issue of extended dissipative analysis for neural networks (NNs) with additive time-varying delays (ATVDs) is examined in this research. Some less conservative sufficient conditions are obtained to ensure the NNs are asymptotically stable and extended dissipative by building the agumented Lyapunov-Krasovskii functional, which is achieved by utilizing some mathematical techniques with improved integral inequalities like auxiliary function-based integral inequalities (gives a tighter upper bound). The present study aims to solve the $H_\infty, L_2 - L_\infty$, passivity and (Q, S, R)-γ-dissipativity performance in a unified framework based on the extended dissipativity concept. Following this, the condition for the solvability of the designed NNs with ATVDs is presented in the form of linear matrix inequalities. Finally, the practicality and effectiveness of this approach were demonstrated through four numerical examples.

Keywords: neural networks; linear matrix inequality; extended dissipative; additive time-varying delay; Lyapunov-Krasovskii functional
Mathematics Subject Classification: 93D20, 92B20

1. Introduction

Neural networks (NNs) have been one of the hottest issues because of their successful applications in various fields, such as pattern recognition, image decryption, optimization issues, and associative memory. These advancements are due to the dynamic nature of NNs, which are expected to remain stable. However, to execute complex tasks, NNs require a large number of neurons, resulting in latency delays in information flow. This delay negatively impacts the overall performance and stability of NNs. As a result, researchers have shown interest in studying delay-dependent stability analysis for NNs over the past few decades [1–9].

Moreover, in the NNs, signals that are transferred from one location to another may come into contact with many network segments while being used in specific practical applications of NNs, such as networked control systems and those that are operated remotely. These segments have the potential to generate consecutive delays with variable qualities as a result of the shifting circumstances of the transmission [10]. When examining the stability of these NNs, a single delay model cannot be used because there are several delays involved. The issue was addressed by proposing the idea of NNs with additive time-varying delays (ATVDs), which was first introduced in [11]. Studying the stability of NNs that have two time-varying delay components is more challenging than those with a single time delay. Moreover, the approach used to analyze the stability of NNs with a single delay cannot be directly applied to those with two ATVD components because the delays are additive. As a result, this discovery has received considerable attention in the industry, with numerous notable publications dedicated to analyzing the stability of NNs that account for ATVDs in the system state. Some examples of such publications include [12–16].

Several stability criteria have been proposed for NNs with delays in recent years, including the improved free-weighting matrix approach [17], Jensen's inequality [18], and other integral inequalities [16]. In one study [16], various stability criteria were compared. Meanwhile, another study [19] focused on global asymptotic stability criteria for complex-valued NNs with leakage delay and ATVDs. Two other studies [12,13] utilized the reciprocally convex combination lemma to develop new stability criteria for NNs with additive delays. Another approach proposed in a paper [20] involved stability analysis and stabilization design for systems with additive delays, which was based on delay-partitioning-based Lyapunov-Krasovskii functionals (LKFs). Lastly, a study [21] explored finite-time control design for NNs with both additive delays and average dwell time.

Recently, researchers have been utilizing a variety of methods of analysis to investigate NNs for their dynamic behavior. The dissipative analysis of NNs with time-varying delays is an example of this type of technique, and it has been investigated in a number of papers [22–25]. In [26], the dissipative system and theory provide a framework for developing and analyzing control systems based on energy-related issues. Because dissipation inequalities constitute a useful tool that possesses distinctive benefits, such as the ability to efficiently attenuate interference; therefore, it is essential to do research on this subject. Extended dissipativity analysis was first developed by Zhang et al. [27]. The study of H_∞ performance, $L_2 - L_\infty$ performance, passivity and (Q, S, R)-γ-dissipativity performance simultaneously yielded an extended dissipativity theory for continuous-time delayed NNs in [28]. The authors introduced the neuron activation function in two cases, which enabled them to modify the weighting matrices and incorporate various performance indices, including H_∞ performance, $L_2 - L_\infty$ performance, passivity, and (Q, S, R)-γ-dissipativity. In [29], the authors proposed an extended dissipativity approach to solve the stabilization problem for a class of uncertain time-delayed NNs. In [30], the same authors extended their previous work by proposing an extended dissipativity-based approach for the stabilization of switched time-delayed NNs. In [31], the authors studied the problem of finite-time extended dissipativity for a class of time-delayed NNs. In [32], the authors focused on the extended dissipativity problem for a class of time-delayed NNs with stochastic perturbations. They developed an extended dissipativity-based fuzzy networked controller to guarantee the H_∞ extended dissipativity of the closed-loop system. Lastly, in [33], the authors proposed an extended dissipativity-based approach for the sample-data control of time-delayed NNs. They utilized a new extended dissipativity concept and developed a stochastic controller that guarantees the finite-time H_∞ extended dissipativity of the closed-loop system. The authors of [16] proposed novel and robust criteria to analyze the delay-dependent stability of uncertain NNs that incorporate two ATVD components. The issues of passivity and passification for NNs with ATVDs are addressed [34]. A study of Markovian jump NNs with ATVDs was conducted in [35]. Moreover, [36] explores asymptotic stability analysis for generalized NNs with ATVDs. However, the existing literature has not yet addressed the improved results regarding the extended dissipativity of NNs with ATVDs. This has also motivated our present research.

Motivated by the preceding discussion, the objective of this study was to explore and address the problem of extended dissipative analysis for NNs incorporating ATVDs. The research presents novel stability criteria that are specifically for NNs with ATVD components, aiming to get less conservative results. The key contributions of this work can be summarized as follows:

(i) The introduction of the concept of ATVD components, denoted as $\delta(t) = \delta_1(t) + \delta_2(t)$, where $\delta_1(t)$ represents the lower bound and $\delta_2(t)$ represents the upper bound of the system's delay.

(ii) The proposal of an innovative method for designing stability criteria suitable for NNs with ATVD components, subject to certain conditions. This approach utilizes techniques based on auxiliary function-based integral inequalities (AFBIIs) and incorporates an augmented LKF that integrates information from the ATVD components.

(iii) The derivation of an improved integral inequality through a parameter transformation approach. This inequality stands apart from existing ones and enables the derivation of tighter bounds for various types of integral quadratic terms. It also contributes to the development of less conservative stability criteria for NNs with ATVD components, surpassing the achievements of previous results.

(iv) The introduction of the concept of extended dissipativity for ATVDs, encompassing diverse performance measures such as $L_2 - L_\infty$, H_∞, passivity, and (Q, S, R)-γ-dissipativity performances of NNs.

(v) The validation of the proposed method is demonstrated through the presentation of four numerical examples. These examples serve to highlight the effectiveness of the method in designing stable NNs with reduced conservatism.

(vi) In addition, to demonstrate the possibility of a benchmark problem, the quadruple tank process system (QTPS) is studied in this work in terms of the NN model.

Notations: The notations throughout this paper are standard, which can be found in [16].

2. System description and preliminaries

Consider the following neural network with ATVDs:

$$\begin{aligned}\dot{\zeta}(t) &= -A\zeta(t) + Bf(\zeta(t)) + Cf(\zeta(t - \delta_1(t) - \delta_2(t)) + u(t),\\ y(t) &= D\zeta(t),\end{aligned}\tag{2.1}$$

where $\zeta(t) \in \mathbb{R}^n$ is a state vector, A is the positive diagonal matrix and B, C, and D are known constant matrices with appropriate dimensions. The function $f(\zeta(t)) \in \mathbb{R}^n$ denotes the neuron activation function, $u(t) \in \mathbb{R}^m$ is the disturbance input belonging to $\mathscr{L}_2 \in [0, \infty)$ and $y(t) \in \mathbb{R}^q$ is the output of the NNs. The delays $\delta_1(t)$ and $\delta_2(t)$ are both non-negative, differentiable functions that satisfy

$$0 \le \delta_1(t) \le \delta_1, \quad 0 \le \delta_2(t) \le \delta_2 \tag{2.2}$$

and

$$0 \le \dot{\delta_1}(t) \le \mu_1 < \infty, \quad 0 \le \dot{\delta_2}(t) \le \mu_2 < \infty, \tag{2.3}$$

where $\delta_1, \delta_2, \mu_1$, and μ_2 are constants. We denote

$$\delta(t) = \delta_1(t) + \delta_2(t), \qquad \delta = \delta_1 + \delta_2, \quad \mu = \mu_1 + \mu_2, \tag{2.4}$$

and μ is supposed as $\mu \le 1$ in this paper.

This article aims to propose novel and less conservative stability criteria for NNs that are described by (2.1) and meet the conditions specified in (2.2) and (2.3). To achieve this objective, we will employ a range of technical lemmas, along with other useful inequalities, in our analysis.

Our approach focuses on developing an improved LKF that can enable us to establish a delay-dependent stability criterion for the NN (2.1). The resulting criterion can overcome the limitations of the existing methods, which are often conservative and may lead to overly restrictive stability conditions.

Assumption 1. For all $\zeta_1, \zeta_2 \in \mathbb{R}$, $\zeta_1 \neq \zeta_2$, the neuron activation function $f_i(\cdot)$ satisfies:

$$\varrho_i^- \le \frac{f_i(\zeta_1) - f_i(\zeta_2)}{\zeta_1 - \zeta_2} \le \varrho_i^+ \quad for\ all\ \ \zeta_1, \zeta_2 \in \mathbb{R}, \quad \zeta_1 \neq \zeta_2.$$

For presentation convenience, the following notations are defined:

$$\begin{aligned}\Delta_1 &= diag\{\varrho_1^-\varrho_1^+, \varrho_2^-\varrho_2^+, ..., \varrho_n^-\varrho_n^+\},\\ \Delta_2 &= diag\{\frac{\varrho_1^- + \varrho_1^+}{2}, \frac{\varrho_2^- + \varrho_2^+}{2}, ..., \frac{\varrho_n^- + \varrho_n^+}{2}\}.\end{aligned}$$

Assumption 2. [28] Matrices Φ_1, Φ_2, Φ_3, and Φ_4 satisfy the following conditions:

1) $\Phi_1 = \Phi_1^T \le 0,\ \Phi_3 = \Phi_3^T > 0,\ \Phi_4 = \Phi_4^T \ge 0,$

2) $(\|\Phi_1\| + \|\Phi_2\|).\|\Phi_4\| = 0.$

Definition 2.1. *[28] For given matrices Φ_1, Φ_2, Φ_3 and Φ_4 satisfying Assumption 2, the NN (2.1) is said to be extended dissipative for any $T_f \geq 0$ and $u(t) \in \mathscr{L}_2[0, \infty)$, under the zero initial state, such that the following inequality holds:*

$$\int_0^{T_f} J(t)dt \geq \sup_{0 \leq t \leq T_f} y^T(t)\Phi_4 y(t),$$

where $J(t) = y^T(t)\Phi_1 y(t) + 2y^T(t)\Phi_2 u(t) + u^T(t)\Phi_3 u(t)$.

Lemma 2.2. *[37] For a positive definite matrix $R > 0$ and a differentiable function $\{x(\alpha)|\alpha \in [a, b]\}$ the following inequality holds:*

$$\begin{aligned}
&\int_a^b x^T(\alpha)Rx(\alpha)d\alpha \geq \frac{1}{(b-a)}\Big(\int_a^b x(\alpha)d\alpha\Big)^T R\Big(\int_a^b x(\alpha)d\alpha\Big) + \frac{3}{b-a}\Theta_1^T R\Theta_1, \\
&\int_a^b \dot{x}^T(\alpha)R\dot{x}(\alpha)d\alpha \geq \frac{1}{(b-a)}\Theta_2^T R\Theta_2 + \frac{3}{(b-a)}\Theta_3^T R\Theta_3,
\end{aligned}$$

where,

$$\begin{aligned}
\Theta_1 &= \int_a^b x(\alpha)d\alpha - \frac{2}{(b-a)}\int_a^b \int_\beta^b x(\alpha)d\alpha d\beta, \\
\Theta_2 &= x(b) - x(a), \\
\Theta_3 &= x(b) + x(a) - \frac{2}{(b-a)}\int_a^b x(\alpha)d\alpha.
\end{aligned}$$

Lemma 2.3. *[38] Let $f_1, f_2, ..., f_N : \mathbb{R}^m \to \mathbb{R}$ have positive values in an open subset $\mathbf{D}$ of $\mathbb{R}^m$ that satisfies*

$$\min_{\{\alpha_i|\alpha_i>0, \Sigma_i\alpha_i=1\}} \Sigma_i \frac{1}{\alpha_i} f_i(t) = \Sigma_i f_i(t) + \max_{g_{j,i}(t)} \Sigma_{i\neq j} g_{i,j}(t)$$

subject to

$$\left\{g_{i,j} : \mathbb{R}^m \to \mathbb{R}, g_{j,i}(t) \equiv g_{i,j}(t), \begin{pmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_i(t) \end{pmatrix} \geq 0\right\}. \tag{2.5}$$

Lemma 2.4. *[39] For the function $\alpha_i(t) \in (0, 1)$ $(i = 1, 2, 3, 4), \Sigma_{i=1}^4 \alpha_i(t) = 1$, vectors $\chi_1(t)$, $\chi_2(t)$, $\chi_3(t)$, $\chi_4(t)$ and a matrix $R_1 > 0$, define the function ϕ as follows:*

$$\phi = \frac{1}{\alpha_1(t)}\chi_1^T(t)R_1\chi_1(t) + \frac{1}{\alpha_2(t)}\chi_2^T(t)R_1\chi_2(t) + \frac{1}{\alpha_3(t)}\chi_3^T(t)R_2\chi_3(t) + \frac{1}{\alpha_4(t)}\chi_4^T(t)R_2\chi_4(t).$$

Assume that there exists a symmetric matrix Z satisfying $\begin{pmatrix} R_1 & Z \\ Z^T & R_2 \end{pmatrix} > 0$ such that the following inequality holds:

$$\phi \geq \eta^T(t)\begin{pmatrix} R_1 & -R_1 & Z & Z \\ * & R_1 & Z & Z \\ * & * & R_2 & -R_2 \\ * & * & * & R_2 \end{pmatrix}\eta(t), \tag{2.6}$$

where $\eta^T(t) = [\chi_1^T(t),\ \chi_2^T(t),\ \chi_3^T(t),\ \chi_4^T(t)]$.

Remark 2.5. *It is important to highlight the significance of the improved integral inequality (referred to as inequality* (2.6)*), which introduces a novel approach to bounding integral quadratic terms with varying time delays. This approach differs from the reciprocally convex combination inequality proposed by Park et al. in [38]. Unlike the previous inequality, there is no requirement to restrict the condition as described in* (2.5) *for the improved integral inequality. Furthermore, as compared to [38], the improved integral inequality* (2.6) *involves fewer free-weighting matrices, while maintaining the same number of scales, denoted as α_i. This reduction in the number of weight matrices indicates that the computational complexity of the improved inequality is lower than that of the inequality in [38]. Additionally, the improved integral inequality* (2.6) *enhances the upper bounds of integral quadratic terms as compared to [38] when considering the same number of scales α_i. This enhancement in the upper bounds can be beneficial in deriving stability results that are less conservative.*

3. Main results

3.1. Extended dissipative analysis

This section introduces a novel approach to derive delay-dependent stability conditions for NNs with ATVDs, which satisfy conditions (2.2) and (2.3). To achieve this, a new class of LKF is proposed and combined with an AFBII technique and an improved integral inequality technique. This combination allows for the analysis of extended dissipativity in the NN described by (2.1).

Theorem 3.1. *For given positive scalars δ_1, δ_2, μ_1 and μ_2, the NNs with ATVDs given by* (2.1) *is extended-dissipative if there exist appropriate symmetric matrices such as $\mathscr{P} = [P_i]_{7\times 7} > 0$, $\mathscr{Q}_i > 0$ (for $i = 1, 2, ..., 7$), $\mathscr{R}_i > 0$ and $\mathscr{S}_i > 0$ (for $i = 1, 2, 3$), as well as matrices $T_{11}, T_{12}, T_{22}, Y_{11}, Y_{12}$ and Y_{22} with the appropriate dimensions and diagonal matrices U and $\bar{U}$. Furthermore, there should exist a scalar γ such that certain linear matrix inequalities hold.*

$$\begin{bmatrix} \mathscr{S}_1 & 0 & T_{11} & T_{12} \\ * & 3\mathscr{S}_1 & T_{12}^T & T_{22} \\ * & * & \mathscr{S}_2 & 0 \\ * & * & * & 3\mathscr{S}_2 \end{bmatrix} \geq 0, \tag{3.1}$$

$$\begin{bmatrix} \mathscr{S}_3 & 0 & Y_{11} & Y_{12} \\ * & 3\mathscr{S}_3 & Y_{12}^T & Y_{22} \\ * & * & \mathscr{S}_3 & 0 \\ * & * & * & 3\mathscr{S}_3 \end{bmatrix} \geq 0, \tag{3.2}$$

$$P_1 - D^T \Phi_4 D \geq 0, \tag{3.3}$$

$$\begin{aligned}
\Psi(\delta_1(t),\ \delta_2(t)) = \Big[& sym\Big\{\Big[e_1^T\ \ \delta_2(t)e_{10}^T + (\delta_2 - \delta_2(t))e_{11}^T\ \ \delta_1(t)e_{12}^T + (\delta_1 - \delta_1(t))e_{13}^T \\
& \delta(t)e_{14}^T + (\delta - \delta(t))e_{15}^T\ \ \delta_1^2 e_{16}^T\ \ \delta_2^2 e_{17}^T\ \ \delta^2 e_{18}^T\Big]\mathscr{P}\Big[\omega^T\ \ e_4^T - e_9^T\ \ e_6^T - e_9^T \\
& e_1^T - e_9^T\ \ \delta_2 e_4^T - \delta_2(t)e_{10}^T - (\delta_2 - \delta_2(t))e_{11}^T\ \ \delta_1 e_6^T - \delta_1(t)e_{12}^T - (\delta_1 - \delta_1(t))e_{13}^T \\
& \delta e_1^T - \delta(t)e_{14}^T - (\delta - \delta(t))e_{15}^T\Big]^T\Big\} + e_1^T(\mathscr{Q}_1 + \mathscr{Q}_2 + \mathscr{Q}_6)e_1 - e_4^T(\mathscr{Q}_1 - \mathscr{Q}_4)e_4 \\
& - e_6^T(\mathscr{Q}_2 - \mathscr{Q}_5)e_6 - e_9^T\mathscr{Q}_3 e_9 - (1-\mu_2)e_5^T\mathscr{Q}_4 e_5 - (1-\mu_1)e_7^T\mathscr{Q}_5 e_7 - (1-\mu) \\
& \times e_8^T(\mathscr{Q}_6 - \mathscr{Q}_3)e_8 + \delta\omega^T(\delta_2\mathscr{S}_1 + \delta_1\mathscr{S}_2 + \delta\mathscr{S}_3)\omega + e_1^T(\delta_2\mathscr{R}_1 + \delta_1\mathscr{R}_2 + \delta\mathscr{R}_3)e_1 \\
& - \Xi_1^T\psi_1\Xi_1 - \Xi_2^T\psi_2\Xi_2 - \frac{1}{\delta_2}Z_1^T\mathscr{R}_1 Z_1 - \frac{3}{\delta_2}Z_2^T\mathscr{R}_1 Z_2 - \frac{1}{\delta_1}Z_3^T\mathscr{R}_2 Z_3 \\
& - \frac{3}{\delta_1}Z_4^T\mathscr{R}_2 Z_4 - \frac{1}{\delta}Z_5^T\mathscr{R}_3 Z_5 - \frac{3}{\delta}Z_6^T\mathscr{R}_3 Z_6 - e_1^T\Delta_1 U e_1 + 2e_1\Delta_2 U e_2 - e_2^T U e_2 \\
& - e_8^T\Delta_1\bar{U}e_8 + 2e_8^T\Delta_2\bar{U}e_3 - e_3^T U e_3 - e_1^T D^T\Phi_1 D e_1 - e_1^T\Phi_2^T e_{19} - e_{19}^T\Phi_3 e_{19}\Big] < 0,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
&\Upsilon_1 = e_4 - e_5,\ \ \Upsilon_2 = e_4 + e_5 - 2e_{10},\ \ \Upsilon_3 = e_5 - e_9, \\
&\Upsilon_4 = e_5 + e_9 - 2e_{11},\ \ \Upsilon_5 = e_6 + e_7,\ \ \ \Upsilon_6 = e_6 + e_7 - 2e_{12}, \\
&\Upsilon_7 = e_7 - e_9,\ \Upsilon_8 = e_7 + e_9 - 2e_{13},\ \ \Upsilon_9 = e_1 - e_8,\ \ \Upsilon_{10} = e_1 + e_8 - 2e_{14}, \\
&\Upsilon_{11} = e_8 - e_9,\ \ \Upsilon_{12} = e_8 + e_9 - 2e_{15},\ \ Z_1 = \delta_2(t)e_{10} + (\delta_2 - \delta(t))e_{11}, \\
&Z_2 = \delta_2(t)e_{10} + (\delta_2 - \delta(t))e_{11} - 2\delta_2 e_{16},\ \ Z_3 = \delta_1(t)e_{12} + (\delta_1 - \delta_1(t))e_{13}, \\
&Z_4 = \delta_1(t)e_{12} + (\delta_1 - \delta_1(t))e_{13} - 2\delta_1 e_{17},\ \ Z_5 = \delta(t)e_{14} + (\delta - \delta(t))e_{15}, \\
&Z_6 = \delta(t)e_{14} + (\delta - \delta(t))e_{15} - 2\delta e_{18}, \\
&e_i = \Big[0_{n\times(i-1)n}\quad I_n\quad 0_{n\times(19-i)n}\Big],\ \ \ i = 1, 2, ..., 19.
\end{aligned}$$

Proof. Choose an LKF candidate for the NN (2.1),

$$V(\zeta(t)) = \sum_{i=1}^{3} V_i(\zeta(t)), \tag{3.5}$$

where

$$\begin{aligned}
V_1(\zeta(t)) &= \eta^T(t)\mathscr{P}\eta(t), \\
V_2(\zeta(t)) &= \int_{t-\delta_1}^{t}\zeta^T(s)\mathscr{Q}_1\zeta(s)ds + \int_{t-\delta_2}^{t}\zeta^T(s)\mathscr{Q}_2\zeta(s)ds + \int_{t-\delta}^{t-\delta(t)}\zeta^T(s)\mathscr{Q}_3\zeta(s)ds \\
&\quad + \int_{t-\delta_1-\delta_2(t)}^{t-\delta_1}\zeta^T(s)\mathscr{Q}_4\zeta(s)ds + \int_{t-\delta_2-\delta_1(t)}^{t-\delta_2}\zeta^T(s)\mathscr{Q}_5\zeta(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t-\delta(t)}^{t}\zeta^T(s)\mathscr{Q}_6\zeta(s)ds+\int_{t-\delta(t)}^{t}f^T(\zeta(s))\mathscr{Q}_7 f(\zeta(s))ds,\\
V_3(\zeta(t)) =\ &\delta\int_{-\delta}^{-\delta_1}\int_{t+\theta}^{t}\dot{\zeta}^T(s)\mathscr{S}_1\dot{\zeta}(s)dsd\theta+\delta\int_{-\delta}^{-\delta_2}\int_{t+\theta}^{t}\dot{\zeta}^T(s)\mathscr{S}_2\dot{\zeta}(s)dsd\theta\\
&+\delta\int_{-\delta}^{0}\int_{t+\theta}^{t}\dot{\zeta}^T(s)\mathscr{S}_3\dot{\zeta}(s)dsd\theta+\int_{-\delta}^{-\delta_1}\int_{t+\theta}^{t}\zeta^T(s)\mathscr{R}_1\zeta(s)dsd\theta\\
&+\int_{-\delta}^{-\delta_2}\int_{t+\theta}^{t}\zeta^T(s)\mathscr{R}_2\zeta(s)dsd\theta+\int_{-\delta}^{0}\int_{t+\theta}^{t}\zeta^T(s)\mathscr{R}_3\zeta(s)dsd\theta,
\end{aligned}
$$

where
$\eta^T(t)=\Big[\zeta^T(t)\ \ \int_{t-\delta}^{t-\delta_1}\zeta^T(s)ds\ \ \int_{t-\delta}^{t-\delta_2}\zeta^T(s)ds\ \ \int_{t-\delta}^{t}\zeta^T(s)ds\ \ \int_{t-\delta}^{t-\delta_1}\int_{\theta}^{t-\delta_1}\zeta(s)dsd\theta$
$\int_{t-\delta}^{t-\delta_2}\int_{\theta}^{t-\delta_2}\zeta(s)dsd\theta\ \int_{t-\delta}^{t}\int_{\theta}^{t}\zeta(s)dsd\theta\Big]$.

Our objective now is to compute the time derivative of each $V_i(\zeta(t))$ term, where $(i=1,...,3)$. To simplify the computations, we introduce the following vectors:

$$
\begin{aligned}
\chi^T(t)=\Big[&\zeta^T(t)\ \ f^T(\zeta(t))\ \ f^T(\zeta(t-\delta(t))\ \ \zeta^T(t-\delta_1)\ \ \zeta^T(t-\delta_1-\delta_2(t))\ \ \zeta^T(t-\delta_2)\ \ \zeta^T(t-\delta_2-\delta_1(t))\\
&\zeta^T(t-\delta(t))\ \ \zeta^T(t-\delta)\ \ \frac{1}{\delta_2(t)}\int_{t-\delta_1-\delta_2(t)}^{t-\delta_1}\zeta^T(s)ds\quad \frac{1}{\delta_2-\delta_2(t)}\int_{t-\delta}^{t-\delta_1-\delta_2(t)}\zeta^T(s)ds\\
&\frac{1}{\delta_1(t)}\int_{t-\delta_2-\delta_1(t)}^{t-\delta_2}\zeta^T(s)ds\quad \frac{1}{\delta_1-\delta_1(t)}\int_{t-\delta}^{t-\delta_2-\delta_1(t)}\zeta^T(s)ds\quad \frac{1}{\delta(t)}\int_{t-\delta(t)}^{t}\zeta^T(s)ds\\
&\frac{1}{\delta-\delta(t)}\int_{t-\delta}^{t-\delta(t)}\zeta^T(s)ds\quad \frac{1}{\delta_2^2}\int_{t-\delta}^{t-\delta_1}\int_{\theta}^{t-\delta_1}\zeta^T(s)dsd\theta\quad \frac{1}{\delta_1^2}\int_{t-\delta}^{t-\delta_2}\int_{\theta}^{t-\delta_2}\zeta^T(s)dsd\theta\\
&\frac{1}{\delta^2}\int_{t-\delta}^{t}\int_{\theta}^{t}\zeta^T(s)dsd\theta\quad u^T(t)\Big],\\
\omega = &Ae_1+Be_2+Ce_3+u(t).
\end{aligned}
$$

Now, we obtain the time derivative of $V(\zeta(t))$:

$$
\begin{aligned}
\dot{V}_1(\zeta(t)) &= 2\eta^T(t)\mathscr{P}\dot{\eta}(t)\\
&=2\Big[e_1^T\ \ \delta_2(t)e_{10}^T+(\delta_2-\delta_2(t))e_{11}^T\ \ \delta_1(t)e_{12}^T+(\delta_1-\delta_1(t))e_{13}^T\\
&\quad\ \delta(t)e_{14}^T+(\delta-\delta(t))e_{15}^T\ \ \delta_1^2e_{16}^T\ \ \delta_2^2e_{17}^T\ \ \delta^2e_{18}^T\Big]\mathscr{P}\dot{\eta}(t)\\
&=2\chi^T(t)\Big[\Big(e_1^T\ \ \delta_2(t)e_{10}^T+(\delta_2-\delta_2(t))e_{11}^T\ \ \delta_1(t)e_{12}^T+(\delta_1-\delta_1(t))e_{13}^T\\
&\quad\ \delta(t)e_{14}^T+(\delta-\delta(t))e_{15}^T\delta_1^2e_{16}^T\ \ \delta_2^2e_{17}^T\ \ \delta^2e_{18}^T\Big)\mathscr{P}\Big(\omega^T\ \ e_4^T-e_9^T\ \ e_6^T-e_9^T\ \ e_1^T-e_9^T\\
&\quad\ \delta_2e_4^T-\delta_2(t)e_{10}^T-(\delta_2-\delta_2(t))e_{11}^T\ \ \delta_1e_6^T-\delta_1(t)e_{12}^T-(\delta_1-\delta_1(t))e_{13}^T\ \ \delta e_1^T-\delta(t)e_{14}^T-(\delta-\delta(t))e_{15}^T\Big)\Big]^T\chi(t).
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\dot{V}_2(\zeta(t)) \leq\ & \zeta^T(t)\mathscr{Q}_1\zeta(t) + \zeta^T(t)\mathscr{Q}_2\zeta(t) + \zeta^T(t)\mathscr{Q}_6\zeta(t) - \zeta^T(t-\delta_1)(\mathscr{Q}_1 - \mathscr{Q}_4)\zeta(t-\delta_1) \\
& - \zeta^T(t-\delta_2)(\mathscr{Q}_2 - \mathscr{Q}_5)\zeta(t-\delta_2) - \zeta^T(t-\delta)\mathscr{Q}_3\zeta(t-\delta) - (1-\mu_2)\zeta^T(t-\delta_1-\delta_2(t)) \\
& \times \mathscr{Q}_4\zeta(t-\delta_1-\delta_2(t)) - (1-\mu_1)\zeta^T(t-\delta_2-\delta_1(t))\mathscr{Q}_5\zeta(t-\delta_2-\delta_1(t)) \\
& - (1-\mu)\zeta^T(t-\delta(t))(\mathscr{Q}_6 - \mathscr{Q}_3)\zeta(t-\delta(t)) + f^T(\zeta(t))\mathscr{Q}_7 f(\zeta(t)) - f^T(\zeta(t-\delta(t)))\mathscr{Q}_7 f(\zeta(t-\delta(t))), \\
=\ & \chi^T(t)\Bigg[e_1^T(\sum_{i=1}^{2}\mathscr{Q}_i + \mathscr{Q}_6)e_1 - e_4^T(\mathscr{Q}_1 - \mathscr{Q}_4)e_4 - e_6^T(\mathscr{Q}_2 - \mathscr{Q}_5)e_6 - e_9^T\mathscr{Q}_3 e_9 - (1-\mu_2)e_5^T\mathscr{Q}_4 e_5 \\
& - (1-\mu_1)e_7^T\mathscr{Q}_5 e_7 - (1-\mu)e_8^T(\mathscr{Q}_6 - \mathscr{Q}_3)e_8\Bigg]\chi(t),
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\dot{V}_3(\zeta(t)) =\ & \delta\delta_2\dot{\zeta}^T(t)\mathscr{S}_1\dot{\zeta}(t) + \delta\delta_1\dot{\zeta}^T(t)\mathscr{S}_2\dot{\zeta}(t) + \delta^2\dot{\zeta}^T(t)\mathscr{S}_3\dot{\zeta}(t) + \delta_2\zeta^T(t)\mathscr{R}_1\zeta(t) + \delta_1\zeta^T(t)\mathscr{R}_2\zeta(t) \\
& \delta\zeta^T(t)\mathscr{R}_3\zeta(t) - \delta\int_{t-\delta}^{t-\delta_1}\dot{\zeta}^T(s)\mathscr{S}_1\dot{\zeta}(s)ds - \delta\int_{t-\delta}^{t-\delta_2}\dot{\zeta}^T(s)\mathscr{S}_2\dot{\zeta}(s)ds - \delta\int_{t-\delta}^{t}\dot{\zeta}^T(s)\mathscr{S}_3\dot{\zeta}(s)ds \\
& - \int_{t-\delta}^{t-\delta_1}\zeta^T(s)\mathscr{R}_1\zeta(s)ds - \int_{t-\delta}^{t-\delta_2}\zeta^T(s)\mathscr{R}_2\zeta(s)ds - \int_{t-\delta}^{t}\zeta^T(s)\mathscr{R}_3\zeta(s)ds.
\end{aligned} \tag{3.8}
$$

Based on the integration mentioned above, we can utilize Lemmas 2.2–2.4 to analyze the following:

$$
-\delta\int_{t-\delta}^{t-\delta_1}\dot{\zeta}^T(s)\mathscr{S}_1\dot{\zeta}(s)ds - \delta\int_{t-\delta}^{t-\delta_2}\dot{\zeta}^T(s)\mathscr{S}_2\dot{\zeta}(s)ds - \delta\int_{t-\delta}^{t}\dot{\zeta}^T(s)\mathscr{S}_3\dot{\zeta}(s)ds,
$$

and

$$
-\int_{t-\delta}^{t}\zeta^T(s)\mathscr{R}_3\zeta(s)ds - \int_{t-\delta}^{t-\delta_1}\zeta^T(s)\mathscr{R}_1\zeta(s)ds - \int_{t-\delta}^{t-\delta_2}\zeta^T(s)\mathscr{R}_2\zeta(s)ds;
$$

this can be managed as follows by applying (3.8) in Lemma 2.2, respectively.

$$
\begin{aligned}
-\delta\int_{t-\delta}^{t-\delta_1}\dot{\zeta}^T(s)\mathscr{S}_1\dot{\zeta}(s)ds \leq & -\frac{\delta}{\delta_1(t)}\begin{pmatrix}\Upsilon_1 \\ \Upsilon_2\end{pmatrix}^T\begin{pmatrix}\mathscr{S}_1 & 0 \\ 0 & 3\mathscr{S}_1\end{pmatrix}\begin{pmatrix}\Upsilon_1 \\ \Upsilon_2\end{pmatrix} \\
& - \frac{\delta}{\delta_1 - \delta_1(t)}\begin{pmatrix}\Upsilon_3 \\ \Upsilon_4\end{pmatrix}^T\begin{pmatrix}\mathscr{S}_1 & 0 \\ 0 & 3\mathscr{S}_1\end{pmatrix}\begin{pmatrix}\Upsilon_3 \\ \Upsilon_4\end{pmatrix},
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
-\delta\int_{t-\delta}^{t-\delta_2}\dot{\zeta}^T(s)\mathscr{S}_2\dot{\zeta}(s)ds \leq & -\frac{\delta}{\delta_2(t)}\begin{pmatrix}\Upsilon_5 \\ \Upsilon_6\end{pmatrix}^T\begin{pmatrix}\mathscr{S}_2 & 0 \\ 0 & 3\mathscr{S}_2\end{pmatrix}\begin{pmatrix}\Upsilon_5 \\ \Upsilon_6\end{pmatrix} \\
& - \frac{\delta}{\delta_2 - \delta_2(t)}\begin{pmatrix}\Upsilon_7 \\ \Upsilon_8\end{pmatrix}^T\begin{pmatrix}\mathscr{S}_2 & 0 \\ 0 & 3\mathscr{S}_2\end{pmatrix}\begin{pmatrix}\Upsilon_7 \\ \Upsilon_8\end{pmatrix},
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
-\delta\int_{t-\delta}^{t}\dot{\zeta}^T(s)\mathscr{S}_3\dot{\zeta}(s)ds \leq & -\frac{\delta}{\delta(t)}\begin{pmatrix}\Upsilon_9 \\ \Upsilon_{10}\end{pmatrix}^T\begin{pmatrix}\mathscr{S}_3 & 0 \\ 0 & 3\mathscr{S}_3\end{pmatrix}\begin{pmatrix}\Upsilon_9 \\ \Upsilon_{10}\end{pmatrix} \\
& - \frac{\delta}{\delta - \delta(t)}\begin{pmatrix}\Upsilon_{11} \\ \Upsilon_{12}\end{pmatrix}^T\begin{pmatrix}\mathscr{S}_3 & 0 \\ 0 & 3\mathscr{S}_3\end{pmatrix}\begin{pmatrix}\Upsilon_{11} \\ \Upsilon_{12}\end{pmatrix},
\end{aligned} \tag{3.11}
$$

where

$$\begin{aligned}
\Upsilon_1 &= \zeta(t-\delta_1) - \zeta(t-\delta_1-\delta_2(t)) = (e_4 - e_5)\chi(t),\\
\Upsilon_2 &= \zeta(t-\delta_1) - \zeta(t-\delta_1-\delta_2(t)) - \frac{2}{\delta_2(t)}\int_{t-\delta_1-\delta_2(t)}^{t-\delta_1}\zeta(s)ds = (e_4 + e_5 - 2e_{10})\chi(t),\\
\Upsilon_3 &= \zeta(t-\delta_1-\delta_2(t)) - \zeta(t-\delta) = (e_5 - e_9)\chi(t),\\
\Upsilon_4 &= \zeta(t-\delta) + \zeta(t-\delta_1-\delta_2(t)) - \frac{2}{\delta_2-\delta_2(t)}\int_{t-\delta}^{t-\delta_1-\delta_2(t)}\zeta(s)ds = (e_5 + e_9 - 2e_{11})\chi(t),\\
\Upsilon_5 &= \zeta(t-\delta_2) - \zeta(t-\delta_2-\delta_1(t)) = (e_6 - e_7)\chi(t),\\
\Upsilon_6 &= \zeta(t-\delta_2) + \zeta(t-\delta_2-\delta_1(t)) - \frac{2}{\delta_1(t)}\int_{t-\delta_2-\delta_1(t)}^{t-\delta_2}\zeta(s)ds = (e_6 + e_7 - 2e_{12})\chi(t),\\
\Upsilon_7 &= \zeta(t-\delta_2-\delta_1(t)) - \zeta(t-\delta) = (e_7 - e_9)\chi(t),\\
\Upsilon_8 &= \zeta(t-\delta_2-\delta_1(t)) + \zeta(t-\delta) - \frac{2}{\delta_1-\delta_1(t)}\int_{t-\delta}^{t-\delta_2-\delta_1(t)}\zeta(s)ds = (e_7 + e_9 - 2e_{13})\chi(t),\\
\Upsilon_9 &= \zeta(t) - \zeta(t-\delta(t)) = (e_1 - e_8)\chi(t),\\
\Upsilon_{10} &= \zeta(t) + \zeta(t-\delta(t)) - \frac{2}{\delta(t)}\int_{t-\delta(t)}^{t}\zeta(s)ds = (e_1 + e_8 - 2e_{14})\chi(t),\\
\Upsilon_{11} &= \zeta(t-\delta(t)) - \zeta(t-\delta) = (e_8 - e_9)\chi(t),\\
\Upsilon_{12} &= \zeta(t-\delta(t)) + \zeta(t-\delta) - \frac{2}{\delta-\delta(t)}\int_{t-\delta}^{t-\delta(t)}\zeta(s)ds = (e_8 + e_9 - 2e_{15})\chi(t).
\end{aligned}$$

By using Lemmas 2.3 and 2.4, it is easy from (3.1) and (3.2) to obtain that

$$\delta\int_{t-\delta}^{t-\delta_1}\dot{\zeta}^T(s)\mathscr{S}_1\dot{\zeta}(s)ds - \delta\int_{t-\delta}^{t-\delta_2}\dot{\zeta}^T(s)\mathscr{S}_2\dot{\zeta}(s)ds - \delta\int_{t-\delta}^{t}\dot{\zeta}^T(s)\mathscr{S}_3\dot{\zeta}(s)ds \le -\Xi_1^T\psi_1\Xi_1 - \Xi_2^T\psi_2\Xi_2, \tag{3.12}$$

with

$$\Xi_1 = \begin{bmatrix}\Upsilon_1 & \Upsilon_2 & \Upsilon_3 & \Upsilon_4 & \Upsilon_5 & \Upsilon_6 & \Upsilon_7 & \Upsilon_8\end{bmatrix},\ \Xi_2 = \begin{bmatrix}\Upsilon_9 & \Upsilon_{10} & \Upsilon_{11} & \Upsilon_{12}\end{bmatrix},$$

$$\psi_1 = \begin{bmatrix}
\mathscr{S}_1 & 0 & -\mathscr{S}_1 & 0 & T_{11} & T_{12} & T_{11} & T_{12}\\
* & 3\mathscr{S}_1 & 0 & -3\mathscr{S}_1 & T_{12}^T & T_{22} & T_{12}^T & T_{22}\\
* & * & \mathscr{S}_1 & 0 & T_{11} & T_{12} & T_{11} & T_{12}\\
* & * & * & 3\mathscr{S}_1 & T_{12}^T & T_{22} & T_{12}^T & T_{22}\\
* & * & * & * & \mathscr{S}_2 & 0 & -\mathscr{S}_2 & 0\\
* & * & * & * & * & 3\mathscr{S}_2 & 0 & -3\mathscr{S}_2\\
* & * & * & * & * & * & \mathscr{S}_2 & 0\\
* & * & * & * & * & * & * & 3\mathscr{S}_2
\end{bmatrix},$$

$$\psi_2 = \begin{bmatrix} \mathscr{S}_3 & 0 & Y_{11} & Y_{12} \\ * & 3\mathscr{S}_3 & Y_{12}^T & Y_{22} \\ * & * & \mathscr{S}_3 & 0 \\ * & * & * & 3\mathscr{S}_3 \end{bmatrix}.$$

On the other hand, with Lemma 2.2, one can get that

$$\begin{aligned} -\int_{t-\delta}^{t-\delta_1} \zeta^T(s)\mathscr{R}_1\zeta(s)ds &\leq -\frac{1}{\delta_2}Z_1^T\mathscr{R}_1Z_1 - \frac{3}{\delta_2}Z_2^T\mathscr{R}_1Z_2, \\ -\int_{t-\delta}^{t-\delta_2} \zeta^T(s)\mathscr{R}_2\zeta(s)ds &\leq -\frac{1}{\delta_1}Z_3^T\mathscr{R}_2Z_3 - \frac{3}{\delta_1}Z_4^T\mathscr{R}_2Z_4, \\ -\int_{t-\delta}^{t} \zeta^T(s)\mathscr{R}_3\zeta(s)ds &\leq -\frac{1}{\delta}Z_5^T\mathscr{R}_3Z_5 - \frac{3}{\delta}Z_6^T\mathscr{R}_3Z_6, \end{aligned}$$

where

$$\begin{aligned} Z_1 &= \Big[\delta_2(t)e_{10} + (\delta_2 - \delta_2(t))e_{11}\Big]\chi(t), \quad Z_2 = \Big[\delta_2(t)e_{10} + (\delta_2 - \delta_2(t))e_{11} - 2\delta_2 e_{16}\Big]\chi(t), \\ Z_3 &= \Big[\delta_1(t)e_{12} + (\delta_1 - \delta_1(t))e_{13}\Big]\chi(t), \quad Z_4 = \Big[\delta_1(t)e_{12} + (\delta_1 - \delta_1(t))e_{13} - 2\delta_1 e_{17}\Big]\chi(t), \\ Z_5 &= \Big[\delta(t)e_{14} + (\delta - \delta(t))e_{15}\Big]\chi(t), \qquad Z_6 = \Big[\delta(t)e_{14} + (\delta - \delta(t))e_{15} - 2\delta e_{18}\Big]\chi(t). \end{aligned}$$

Furthermore, on account of Assumption 1, the following inequalities hold:

$$\begin{bmatrix} \zeta(t) \\ f(\zeta(t)) \end{bmatrix}^T \begin{bmatrix} \Delta_1 U & -\Delta_2 U \\ * & U \end{bmatrix} \begin{bmatrix} \zeta(t) \\ f(\zeta(t)) \end{bmatrix} \leq 0, \tag{3.13}$$

$$\begin{bmatrix} \zeta(t-\delta(t)) \\ f(\zeta(t-\delta(t))) \end{bmatrix}^T \begin{bmatrix} \Delta_1 \bar{U} & -\Delta_2 \bar{U} \\ * & \bar{U} \end{bmatrix} \begin{bmatrix} \zeta(t-\delta(t)) \\ f(\zeta(t-\delta(t))) \end{bmatrix} \leq 0. \tag{3.14}$$

According to (3.6)–(3.14) and $\dot{V}(\zeta(t)) \leq 0$, we now define the performance index:

$$\begin{aligned} J(t) &= y^T(t)\Phi_1 y(t) + 2y^T(t)\Phi_2 u(t) + u^T(t)\Phi_3 u(t), \\ &= \zeta^T(t)D^T\Phi_1 D\zeta(t) + 2\zeta^T(t)D^T\Phi_2 u(t) + u^T(t)\Phi_3 u(t). \end{aligned} \tag{3.15}$$

Then, it follows that

$$\dot{V}(\zeta(t)) - J(t) \leq \chi^T(t)\Psi\chi(t) < 0, \tag{3.16}$$

where $J(t) = y^T(t)\Phi_1 y(t)$.

In order to prove that the NN (2.1) are extended-dissipative, according to Definition 2.1, we need to show that:

$$\int_0^t J(s)ds \geq V(\zeta(t)) \tag{3.17}$$

holds for matrices Φ_1–Φ_4 with the initial condition $V(0) = 0$. Using the zero initial condition, it can be proven that

$$V(\zeta(t)) \geq \zeta^T(t)P_1\zeta(t) > 0. \tag{3.18}$$

We also have

$$\int_0^t J(s)ds \geq \zeta^T(t)P_1\zeta(t). \tag{3.19}$$

To satisfy (3.16) and thus prove that the NN (2.1) are extended-dissipative, the following inequality must hold:

$$\int_0^{T_f} J(t)dt - \sup_{0\leq t\leq T_f} y^T(t)\Phi_4 y(t) \geq 0. \tag{3.20}$$

The criteria for extended dissipativity given in Assumption 2 states the following:
(i) $\Phi_4 = 0$ when the H_∞ performance, the passivity and the (Q, S, R)-γ-dissipativity conditions are satisfied.

$$\int_0^{T_f} J(t)dt = \sup_{0\leq t\leq T_f} y^T(s)\Phi_4 y(s) \quad \text{for any } \mathrm{T_f} \geq 0.$$

(ii) When $\Phi_4 > 0$, the $L_2 - L_\infty$ performance condition is satisfied. We obtain $\Phi_1 = 0$, $\Phi_2 = 0$ and $\Phi_3 > 0$ from Assumption 2. Then, it can be shown that

$$\int_0^t J(s)ds > 0, \tag{3.21}$$

and for any $t \geq 0$, $T_f \geq 0$, $T_f \geq t$, $0 \leq t \leq T_f$, for all $t \in [0, T_f]$, we have

$$\int_0^{T_f} J(s)ds > \int_0^t J(s)ds \geq \zeta^T(t)P_1\zeta(t) > 0;$$

from (3.3), we get

$$y^T(t)\Phi_4 y(t) = \zeta^T(t)D^T\Phi_4 D\zeta(t). \tag{3.22}$$

Using these inequalities, we can obtain

$$\int_0^{T_f} J(s)ds \geq \zeta^T(t)P_1\zeta(t) \geq \zeta^T(t)D^T\Phi_4 D\zeta(t). \tag{3.23}$$

This ultimately leads to the inequality

$$\int_0^{T_f} J(t)dt - \sup_{0\leq t\leq T_f} y^T(t)\Phi_4 y(t) \geq 0, \tag{3.24}$$

as required by Definition 2.1. Thus, we can conclude that the NN (2.1) is extended-dissipative.

Remark 3.2. *The inequality* (3.4) *is dependent on the time-varying delays* $\delta_1(t)$ *and* $\delta_2(t)$. *In Theorem 3.1, we consider the upper bounds of these two time-varying delays in order to calculate the LMI condition based on* (3.4) *with the following conditions:* $\Psi(0,\ \delta_2)$, $\Psi(\delta_1,\ 0)$, $\Psi(\delta_1,\ \delta_2)$, *and* $\Psi(0,\ 0)$.

Remark 3.3. *[28] Definition 2.1 defines the concept of extended dissipativity, which encompasses several commonly used performance indices when the weighting matrices are set accordingly. This flexibility allows for a broad range of systems to be analyzed using the extended dissipativity framework, making it a valuable tool for system analysis and design:*
(1) $L_2 - L_\infty$ *performance:* $\Phi_1 = 0$, $\Phi_2 = 0$, $\Phi_3 = \gamma^2 I$ *and* $\Phi_4 = I$;
(2) H_∞ *performance:* $\Phi_1 = -I$, $\Phi_2 = 0$, $\Phi_3 = \gamma^2 I$ *and* $\Phi_4 = 0$;
(3) Passivity performance: $\Phi_1 = 0$, $\Phi_2 = I$, $\Phi_3 = \gamma I$ *and* $\Phi_4 = 0$;
(4) (Q, S, R)-γ*-dissipativity performance:* $\Phi_1 = Q$, $\Phi_2 = S$, $\Phi_3 = R - \gamma I$ *and* $\Phi_4 = 0$.

Remark 3.4. *We can further investigate the effectiveness of our stability criteria by examining a special case of NNs with ATVDs described by* (2.1). *Specifically, we will consider a simplified scenario where the system can be described by an NN without the input* ($u(t) = 0$) *and without output* ($y(t) = 0$) *terms. In this case, the NN is modeled using a single equation of the form*

$$\dot{z}(t) = -A\zeta(t) + Bf(\zeta(t)) + Cf(\zeta(t - \delta_1(t) - \delta_2(t))). \tag{3.25}$$

By analyzing this simplified case, we employ the same proof line in Theorem 3.1, and we can gain insights into the performance of our stability criteria and evaluate their conservativeness.

Remark 3.5. *To address the issue of less conservative stability conditions, we drew inspiration from the concepts presented in [28, 39]. Specifically, we incorporated single- and double-integral terms with the AFBII technique. Moreover, we combined these with improved integral inequalities within the framework of the LMIs defined by* (3.1)–(3.4) *during the computation of the LKF* $V(\zeta(t))$. *By employing the improved integral inequality technique described in Theorem 3.1, we can achieve further enhancements in the stability criterion. It is worth noting that the inclusion of an augmented LKF, an AFBII and the utilization of the improved integral inequality technique can yield less conservative results compared to other methodologies [12, 14, 15, 17, 34–36, 40]. This observation is easily verified by referring to Tables 6 and 7.*

Remark 3.6. *For NNs with ATVDs, constructing the LKF* (3.5) *is challenging. The most important feature of the derived results of this paper is constructing the novel LKF* (3.5) *with a quadratic and augmented term in* $V_1(\zeta(t))$, *which is very challenging when the goal is to obtain less conservative results than the results presented in the literature. More specifically to improve the feasible region for the corresponding system, by taking the states with the available information of LKF* (3.5) *as* $\frac{1}{\delta_2(t)}\int_{t-\delta_1-\delta_2(t)}^{t-\delta_1}\zeta^T(s)ds$, $\frac{1}{\delta_2-\delta_2(t)}\int_{t-\delta}^{t-\delta_1-\delta_2(t)}\zeta^T(s)ds$, $\frac{1}{\delta_1(t)}\int_{t-\delta_2-\delta_1(t)}^{t-\delta_2}\zeta^T(s)ds$, $\frac{1}{\delta_1-\delta_1(t)}\int_{t-\delta}^{t-\delta_2-\delta_1(t)}\zeta^T(s)ds$, $\frac{1}{\delta(t)}\int_{t-\delta(t)}^{t}\zeta^T(s)ds$ *and* $\frac{1}{\delta-\delta(t)}\int_{t-\delta}^{t-\delta(t)}\zeta^T(s)ds$, *the stability conditions in Theorem 3.1 sufficiently utilize more information on state variables, which can yield less conservative results than the existing ones.*

4. Numerical examples

In this section, we will give four examples to illustrate the correctness and effectiveness of the proposed method

Example 4.1. *Consider the NNs with ATVDs represented by* (2.1) *with the following parameters*

$$A = \begin{bmatrix} 3.5 & 0 \\ 0 & 4 \end{bmatrix},\ B = \begin{bmatrix} 1.1 & 0.2 \\ -0.1 & 1.3 \end{bmatrix},\ C = \begin{bmatrix} 0.6 & 0.7 \\ 0.3 & -0.2 \end{bmatrix},$$
$$D = \begin{bmatrix} -0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix};$$

the given activation function is $f(\zeta(t)) = 0.1\tanh(\zeta(t))$, *and the values are* $\varrho_1^- = \varrho_2^- = 0$ *and* $\varrho_1^+ = \varrho_2^+ = 1$ *which satisfy Assumption 1; the time-varying delays are* $\mu_1 = 0.1$ *and* $\mu_2 = 0.2$. *By utilizing Theorem 3.1, we compute the maximum upper bound of* δ_2 *for a fixed* $\delta_1 = 1$. *To establish the extended dissipative conditions for the NN* (2.1), *including passivity,* $L_2 - L_\infty$ *performance,* (Q, S, R)-γ*-dissipativity, and* H_∞ *performance, we used the MATLAB LMI toolbox to solve the corresponding LMIs in Theorem 3.1. When examining the extended dissipative properties of the NN* (2.1), *we focus on the weighting matrices* Φ_1–Φ_4. *By establishing these conditions, we can ensure that the system remains stable and achieves the desired performance in the presence of ATVDs.*

$L_2 - L\infty$ ***Performance****: We conducted an analysis of the* $L_2 - L\infty$ *performance of the NN* (2.1), *with parameters* $\Phi_1 = 0$, $\Phi_2 = 0$, $\Phi_3 = \gamma^2 I$ *and* $\Phi_4 = I$. *To determine the feasibility of the problem, we utilized the LMIs in Theorem 3.1 and the MATLAB LMI toolbox, and we calculated the minimum* γ *on different* δ_2 *as per Theorem 3.1. The results are presented in Table 3. The state trajectories of the NNs with given parameters* $\mu_1 = 0.1$, $\mu_1 = 0.15$, $\mu_2 = 0.2$ *and* $\mu_2 = 0.25$ *were observed and are plotted in Figures 1 and 2. It was observed that the system was able to maintain stability and converge to zero, demonstrating the* $L_2 - L_\infty$ *performance under the aforementioned parameter values. Based on the simulation results, the proposed approach was deemed feasible and efficient.*

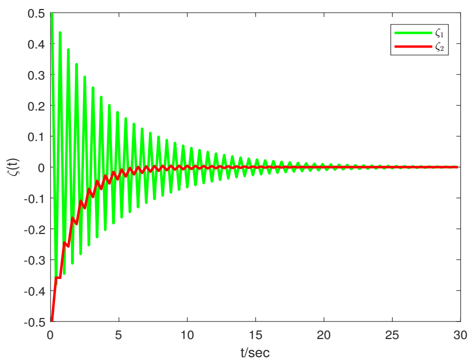

Figure 1. State trajectory of $L_2 - L_\infty$ performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.1$ and $\mu_2 = 0.2$ for Example 4.1.

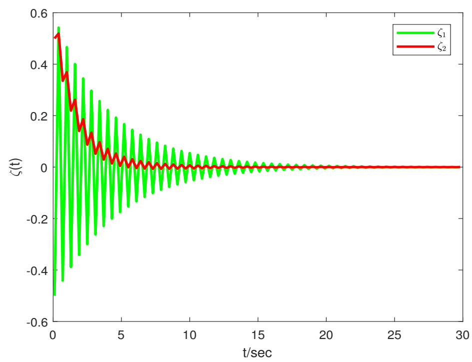

Figure 2. State trajectory of $L_2 - L_\infty$ performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.15$ and $\mu_2 = 0.25$ for Example 4.1.

H∞ ***Performance:*** *We examined the H∞ performance of the* (2.1)*, with parameters* $\Phi_1 = -I$, $\Phi_2 = 0$, $\Phi_3 = \gamma^2 I$ *and* $\Phi_4 = 0$*. The feasibility of the problem was easily obtained using the LMIs described in Theorem 3.1, and the minimum* γ *on different* δ_2 *was calculated as per Theorem 3.1. The results are presented in Table 4. To gain further insight into the performance of the system, simulations were conducted; the subsequent evolution of the state response curves with respect to the given parameters* $\mu_1 = 0.1$, $\mu_1 = 0.15$, $\mu_2 = 0.2$ *and* $\mu_2 = 0.25$ *are plotted in Figure 3. Additionally, the state response of the NNs is shown in Figure 4. It was observed that the system performed well under the aforementioned parameters.*

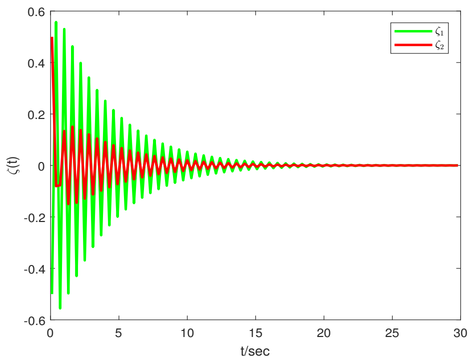

Figure 3. State trajectory of H_∞ performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.1$ and $\mu_2 = 0.2$ for Example 4.1.

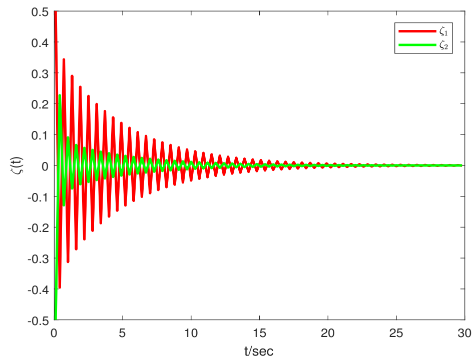

Figure 4. State trajectory of H_∞ performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.15$ and $\mu_2 = 0.25$ for Example 4.1.

Passivity Performance: *The aim of this study was to examine the passivity performance of NNs described by* (2.1)*, with parameters* $\Phi_1 = 0$, $\Phi_2 = I$, $\Phi_3 = \gamma$, *and* $\Phi_4 = 0$*. The feasibility of the problem was easily obtained using the LMIs described in Theorem 3.1, and the upper bound on* δ_2 *was calculated as per Theorem 3.1. The results are presented in Table 1. To validate the results, simulations were conducted using the MATLAB LMI toolbox; the resulting state responses under given parameters* $\mu_1 = 0.1$, $\mu_1 = 0.15$, $\mu_2 = 0.2$ *and* $\mu_2 = 0.25$ *were plotted in Figure 5. Additionally, the performance of the inputs that converged to zero is demonstrated in Figure 6. It was observed that the passivity performance of both figures was consistent with the available parameters.*

Table 1. Allowable upper bounds of δ_2 for passivity analysis and different values of δ_1 with fixed $\mu_1 = 0.1$ and $\mu_2 = 0.2$.

δ_1	0.1	0.2	0.3	0.5	1
δ_2	3.2688	2.8642	2.6342	2.3002	1.8720

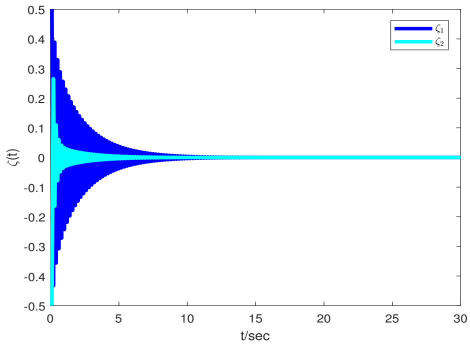

Figure 5. State trajectory of passivity performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.1$ and $\mu_2 = 0.2$ for Example 4.1.

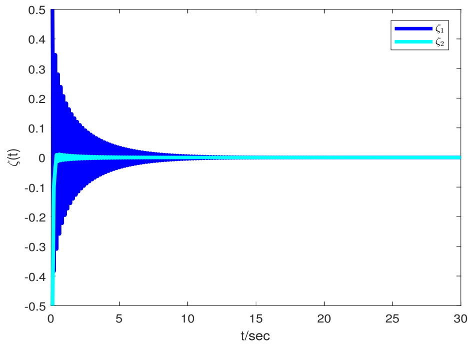

Figure 6. State trajectory of passivity performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.15$ and $\mu_2 = 0.25$ for Example 4.1.

(Q, S, R)-γ-dissipativity: *The parameters used in this study were $\Phi_1 = Q$, $\Phi_2 = S$, $\Phi_3 = R - \gamma I$, and $\Phi_4 = 0$, where Q, S and R denote matrices with specific values: $Q = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$, $S = \begin{bmatrix} 0.2 & 0 \\ 0.4 & 0.25 \end{bmatrix}$, $R = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}$. By solving the LMIs in Theorem 3.1 with these parameters, the dissipativity performance was determined to be $\gamma = 0.08$, and the upper bound on δ_2 was calculated as per Theorem 3.1. These results are presented in Table 2. To validate the results, simulations were conducted using the MATLAB LMI toolbox; the resulting state trajectories under given initial conditions are plotted in Figures 7–10. The simulation results were also explored for the initial condition $[-5;\ 5]^T$ with different values of μ_1 and μ_2. It was observed that the state trajectories converged to zero, indicating that the (Q, S, R)-γ-dissipativity performance requirement was satisfied.*

Table 2. Allowable upper bounds of δ_2 for (Q, S, R)-γ-dissipativity and different values of δ_1 with fixed $\mu_1 = 0.1$ and $\mu_2 = 0.2$.

δ_1	0.1	0.2	0.3	0.5	1
δ_2	2.6124	2.3984	2.1303	1.8816	1.5420

Table 3. Minimum $L_2 - L_\infty$ performance of γ given different values of δ_2 with fixed $\delta_1 = 0$, $\mu_1 = 0.1$ and $\mu_2 = 0.2$.

δ_2	0.1	0.2	0.3	0.4	0.5
γ	0.3488	0.3645	0.3908	0.4203	0.4821

Table 4. Minimum H_∞ performance γ with different values of δ_2 and fixed $\delta_1 = 0$, $\mu_1 = 0.1$ and $\mu_2 = 0.2$.

δ_2	0.1	0.2	0.3	0.4	0.5
γ	0.2531	0.3016	0.4505	0.4620	0.5199

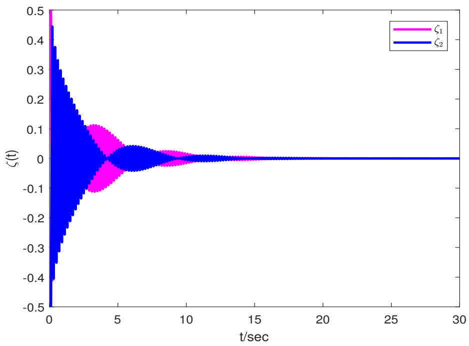

Figure 7. State trajectory of (Q, S, R)-γ-dissipativity performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.1$ and $\mu_2 = 0.2$ for Example 4.1.

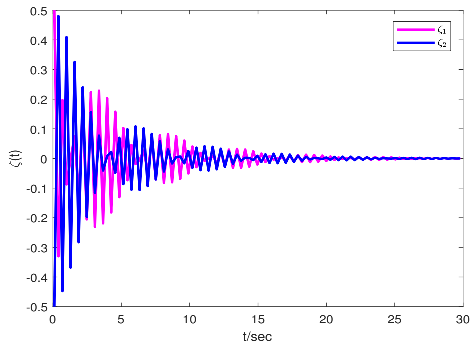

Figure 8. State trajectory of (Q, S, R)-γ-dissipativity performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.15$ and $\mu_2 = 0.25$ for Example 4.1.

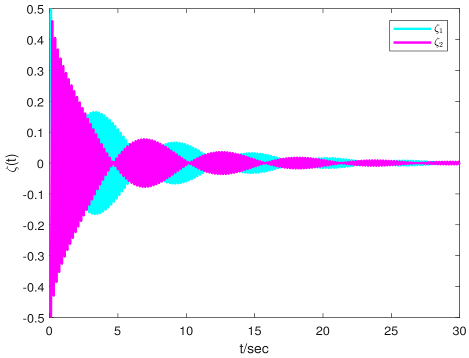

Figure 9. State trajectory of (Q, S, R)-γ-dissipativity performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.2$ and $\mu_2 = 0.3$ for Example 4.1.

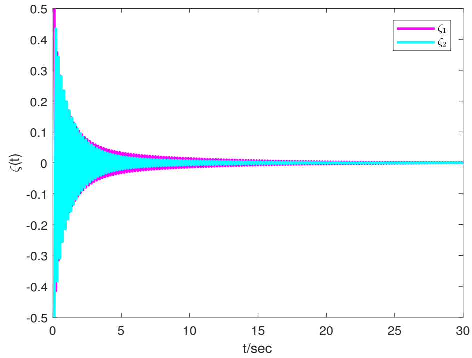

Figure 10. State trajectory of (Q, S, R)-γ-dissipativity performance of $\zeta_1(t)$ and $\zeta_2(t)$ with $\mu_1 = 0.25$ and $\mu_2 = 0.35$ for Example 4.1.

Example 4.2. *For the purpose of this analysis, we will focus on NNs given by* (3.25) *that incorporate ATVDs, with matrix values as follows:*

$$A = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix}, \ B = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$

$$\varrho_1^+ = 0.1137,\ \varrho_2^+ = 0.1279,\ \varrho_3^+ = 0.7994,\ \varrho_4^+ = 0.4480,\ \ \varrho_1^- = \varrho_2^- = \varrho_3^- = \varrho_4^- = 0.$$

To investigate the asymptotic stability of NNs given by (3.25) *with ATVDs, we employed the Matlab LMI toolbox and determined the feasibility of conditions* (3.1)–(3.4)*. Specifically, we solved Theorem 3.1 to obtain the maximum allowable upper bounds (MAUBs) of δ_2 for different values of δ_1, assuming that $\mu_1 = 0.1$ and $\mu_2 = 0.8$. We compared our proposed method with existing approaches [34, 36], and the results are shown in Table 5. The maximum upper bound of δ_2 obtained by Theorem 3.1 is determined for different values of δ_1. Our approach yielded significantly better results than those in [34, 36], indicating that our stability condition is less conservative and more effective. We set the initial condition as $\zeta(0) = [-0.5, 0.3, 0.5, -0.3]^T$; the simulation results of the state responses are shown in Figures 11 and 12 for different parameter values. Our analysis indicates that the NN* (3.25) *with ATVDs is asymptotically stable.*

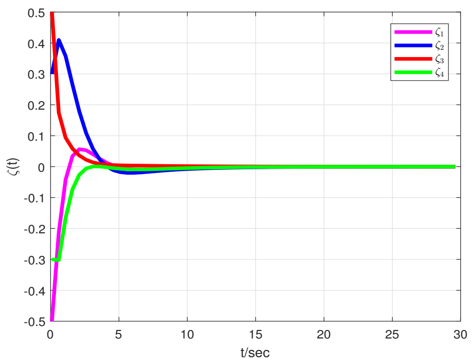

Figure 11. State responses of NNs (3.25) with $\mu_1 = 0.1$ and $\mu_2 = 0.8$ for Example 4.2.

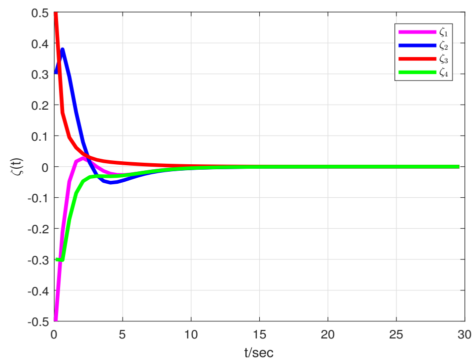

Figure 12. State responses of NNs (3.25) with $\mu_1 = 0.3$ and $\mu_2 = 0.6$ for Example 4.2.

Table 5. MAUBs of δ_2 for different values of δ_1 with fixed $\mu_1 = 0.1$ and $\mu_2 = 0.8$.

Method	1.0	1.2	1.5
[36]($\delta = 0, 1$)	1.7308	1.5245	1.2236
[34]	3.258	3.024	2.678
Remark 3.4	3.8507	3.3196	3.0735

Example 4.3. *For the purpose of this analysis, we will focus on NNs described by* (3.25) *that incorporate ATVDs, with matrix values as follows:*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ C = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix},\ \varrho_1^- = \varrho_2^- = 0,\ \ \varrho_1^+ = 0.4,\ \varrho_2^+ = 0.8.$$

To investigate the asymptotic stability of NNs with ATVDs, we employed the Matlab LMI toolbox and determined the feasibility of conditions (3.1)–(3.4).

Our purpose was to calculate the MAUBs of δ_1 and δ_2 when the other is known, below which the NNs (3.25) *is asymptotically stable. For δ_1 = 0.8, 1.0, 1.2, by using the Matlab LMI toolbox to solve the LMIs in Remark 3.4, it is concluded that system* (3.25) *is asymptotically stable while δ_2 is up to 3.7044, 3.3346, 2.4920, respectively. Similarly, for δ_2 = 0.8, 1.0, 1.2, the corresponding δ_1 is up to 3.6326, 3.3684, 2.8029, respectively. Furthermore, to show the reduced conservatism of the derived result in this paper, the MAUB δ_2 is obtained as in [12, 14, 15, 17, 34–36, 40] and Remark 3.4 for different values of δ_1 and the MAUB δ_1 is obtained as in [12, 34, 40] and Remark 3.4 for different values of δ_2, which are listed in Tables 6 and 7, respectively.*

Table 6. MAUBs of δ_2 for different values of δ_1 with $\mu_1 = 0.7$ and $\mu_2 = 0.1$.

δ_1	[40]	[12]	[14]	[15]	[34]	[35]	[17]	[36]($\delta = 0, 1$)	Remark 3.4
0.8	1.5666	1.9528	2.0164	1.9666	2.2448	2.3547	2.5680	2.6160	3.7044
1	1.3668	1.7992	1.8203	1.8351	1.9642	2.0053	2.3678	2.4160	3.3346
1.2	1.1664	1.6441	1.6197	1.6803	1.8591	1.9217	2.1678	2.2160	2.4920

Table 7. MAUBs of δ_1 for different values of δ_2 with $\mu_1 = 0.7$ and $\mu_2 = 0.1$.

δ_2	[40]	[12]	[34]	Remark 3.4
0.8	2.6928	2.7248	2.8545	3.6326
1	2.2389	2.3325	2.4856	3.3684
1.2	2.0639	2.2187	2.4579	2.8029

Example 4.4. *In this example, we will demonstrate the practical application of the proposed result using a real-world model known as the QTPS, depicted in Figure 13. The use of NNs in this context extends beyond biological models and includes practical models like the QTPS. The overall*

physical model of the system is governed by a mathematical equation, which was initially studied by Johansson [41]. The QTPS comprises four interconnected water tanks, two water pumps and two valves. The primary objective is to control the water level in the two lower tanks by utilizing the two pumps. The inputs to the system are the voltages (v_1 and v_2) applied to Pumps 1 and 2, respectively. The outputs are measured by the water levels (h_1 and h_2) in Tanks 1 and 2. Tanks 1 and 2 were positioned below Tanks 3 and 4, allowing water to flow into them through the force of gravity. Therefore, in the context of practical real-world applications, the four-tank water pumping system can be effectively modeled as an NN. Previous studies [41–44] have proposed state-space equations for this four-tank system, demonstrating the application of NNs in modeling its behavior. These equations can be expressed as follows:

$$\dot{\widehat{x}}(t) = \widehat{A}_0(\widehat{x}(t)) + \widehat{A}_1(\widehat{x}(t-\tau_1)) + \widehat{B}_0(\widehat{u}(t-\tau_2)) + \widehat{B}_1(\widehat{u}(t-\tau_3)), \tag{4.1}$$

where

$$\begin{aligned}
\widehat{A}_0 &= \begin{bmatrix} -0.0021 & 0 & 0 & 0 \\ 0 & -0.0021 & 0 & 0 \\ 0 & 0 & -0.0424 & 0 \\ 0 & 0 & 0 & -0.0424 \end{bmatrix}, \\
\widehat{A}_1 &= \begin{bmatrix} 0 & 0 & 0.0424 & 0 \\ 0 & 0 & 0 & 0.0424 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\widehat{B}_0 &= \begin{bmatrix} 0.1113\rho_1 & 0 & 0 & 0 \\ 0 & 0.1042(1-\rho_2) & 0 & 0 \end{bmatrix}, \\
\widehat{B}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0.1113(1-\rho_1) \\ 0 & 0 & 0.1042(1-\rho_2) & 0 \end{bmatrix}, \\
\rho_1 &= 0.333,\ \rho_2 = 0.307,\ \widehat{u} = \widehat{K}\widehat{x}(t), \\
\widehat{K} &= \begin{bmatrix} -0.1609 & -0.1765 & -0.0795 & -0.2073 \\ -0.1977 & -0.1579 & -0.2288 & -0.0772 \end{bmatrix}.
\end{aligned}$$

Another control problem of our interests is obtained by adding transport delays $\delta(t) = \delta_1(t) + \delta_2(t)$ by delaying the inlet of incoming water into the tanks. Hence, the proposed approach has been used to study this problem here. In this work mainly focuses on transporting time delay signals between valves and tanks being time-varying. For simplicity, it was assumed that $\tau_1 = 0$, $\tau_2 = 0$ and $\tau_3 = \delta(t)$ (since $\delta(t) \leq \delta$). In this example, the control input $\widehat{u}(t)$ indicates the amount of water pumped. Therefore, it is naturally a nonlinear function and can be written as follows:

$$\begin{aligned}
&\widehat{u}(t) = \widehat{K}\widehat{f}(\widehat{x}(t)), \\
&\widehat{f}(\widehat{x}(t)) = [\widehat{f}_1(\widehat{x}_1(t)), ..., \widehat{f}_4(\widehat{x}_4(t))]^T, \\
&\widehat{f}_i(\widehat{x}_i(t)) = 0.1(|\widehat{x}_i(t) + 1| - |\widehat{x}_i(t) - 1|), \qquad i = 1, 2, ...4.
\end{aligned}$$

The four-tank system (4.1) *can be rewritten to the form of system* (3.25) *as follows:*

$$\dot{\zeta}(t) = -A\zeta(t) + Bf(\zeta(t)) + Cf(\zeta(t-\delta(t))), \tag{4.2}$$

where $A = \widehat{A_0} - \widehat{A_1}$, $B = \widehat{B_0}\widehat{K}$, $C = \widehat{B_1}\widehat{K}$. *In addition,* $\Delta_1 = diag\{0,0,0,0\}$ *and* $\Delta_2 = diag\{0.1,0.1,0.1,0.1\}$ *with* $\delta_1 = 1.5$, $\delta_2 = 3.1$, $\mu_1 = 0.1$ *and* $\mu_2 = 0.5$. *Using the MATLAB LMI toolbox and solving the inequalities in Theorem 3.1 applicable to Remark 3.4, we were able to obtain a feasible solution; Figure 14 shows the state trajectories of the system converging to zero with an initial state* $[-0.5, 0.5, 0.5, -0.5]^T$, *which leads to the conclusion that QTPS* (4.2) *is stable.*

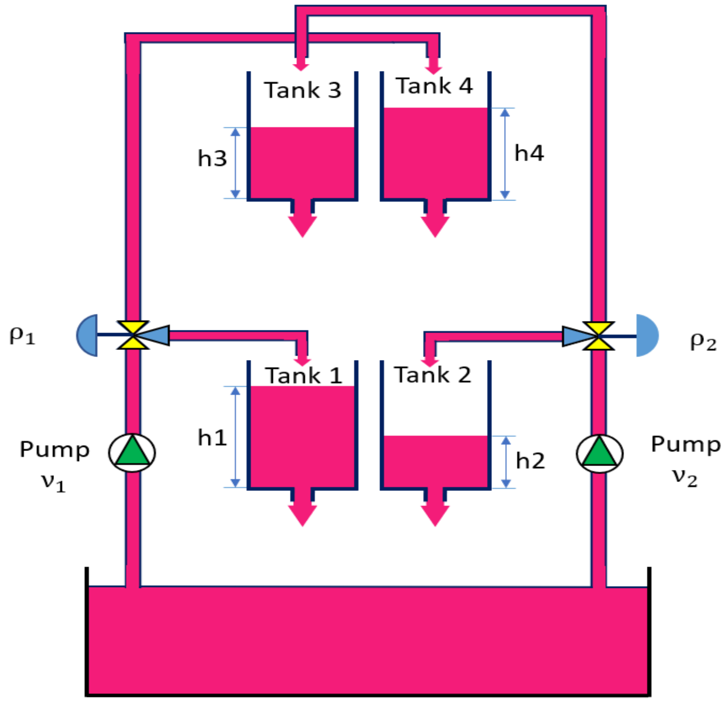

Figure 13. Schematic representation of the QTPS.

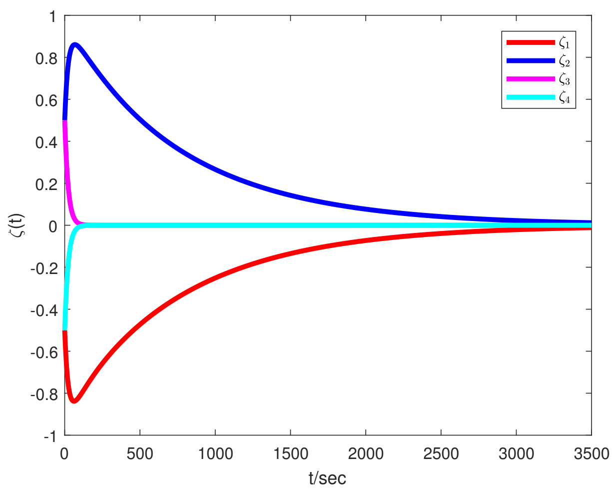

Figure 14. State trajectory of the system (4.2) in Example 4.4.

5. Conclusions

In this paper, we studied improved extended dissipativity performance of NNs that have ATVDs. We proposed delay-dependent stability criteria for the NNs by using a more general and augmented type of LKF, as well as the AFBII technique and improved integral inequality. Our extended dissipativity criteria take into account the relationship between the ATVDs and their upper delay bounds. To demonstrate the effectiveness of our approach, we provided four numerical examples. The results show that our proposed method leads to less conservatism compared to some existing methods. Additionally, the proposed approach has been validated through numerical simulations of a benchmark problem that incorporates ATVDs. In future work, we plan to explore more advanced technologies to design controllers with even less conservatism. Furthermore, we aim to extend our results to more realistic systems, such as delayed nonlinear switched systems and fuzzy switched network systems.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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