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Article

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Abstract: The aim of this paper is to define the generalized discrete proportional derivative (GDPD) and illustrate the application of the Leibniz theorem, the binomial expansion, and Montmort's formulas in the context of the generalized discrete proportional case. Furthermore, we introduce the generalized discrete proportional Laplace transform and determine the GDPLT of various functions using the inverse operator. The results obtained are showcased through relevant examples and validated using MATLAB.

1. Introduction

Undoubtedly, mathematics holds immense value across various disciplines, including physics, economics, and engineering [1–8]. The concept of a difference equation becomes essential when describing the evolution of a phenomenon over time. In the works of [9,10], the discrete case of the differential operator Δ is precisely defined as

$$\Delta f(\hbar) = f(\hbar + 1) - f(\hbar), \qquad \hbar \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}.$$

The discrete cases of derivatives (both real and complex order) and integrals can be defined using discrete fractional calculus, extending the principles of (integer-order) differential calculus [11–13]. Specifically, fractional derivatives in this context are synonymous with Riemann–Liouville fractional derivatives (RLFDs), and their definitions have been explored in various ways in [14–17].

Likewise, discrete fractional derivatives can be defined through various approaches. In the realm of difference calculus, particularly fractional difference calculus [10,18,19], one encounters the forward h-difference operator Δ_h, which is defined as

$$h\Delta_h f(x) = f(x + h) - f(x), \quad x \in [0, \infty), \ h \in (0, \infty).$$

In 1989, Miller and Ross introduced the discrete counterpart of the RLFD, elucidating numerous characteristics of the fractional difference operator [11,20]. This work not only involved proposing the discrete analogue but also encompassed the definition of various generalized difference operators. We further extended our contributions by deriving exact solutions and presenting numerical solutions for a variety of functions [21–23]. Notably, we introduced the discrete variant of the generalized proportional derivative, known as the generalized proportional delta operator.

In the study presented in [24], the authors investigated lower regularized incomplete gamma functions, showcasing their utility in demonstrating the existence and uniqueness of solutions for fractional differential equations involving nonlocal fractional derivatives. Exploring a new frontier, [25] delves into the realm of solutions for a distinct category of nonlinear generalized proportional fractional differential inclusions. This exploration encompasses scenarios where the right-hand side incorporates a Caratheodory-type multi-valued nonlinearity, extending the analysis to infinite intervals. Furthermore, [26] introduces a comprehensive two-step design approach aimed at implementing a fractional-order proportional integral controller tailored for a specific class of fractional-order plant models. Motivated by the preceding discussions and incentives, the principal objective of this paper is to delve into the concept of the GDPD. This investigation involves demonstrating the practical applicability of Leibniz's theorem, binomial expansion, and Montmort's formulas within the generalized discrete proportional setting.

Expanding on prior research, we have extended the discrete form of the generalized proportional differential operator. Its inverse has been skillfully employed to derive fundamental formulas, enabling the computation of closed-form solutions and numerical solutions for distinct categories of finite and infinite series within the domain of number theory. Remarkably, the application of this operator to advance the theoretical foundations of the discrete version remains unparalleled in the existing literature.

A clear distinction often arises between numerical solutions and closed-form solutions in prevalent methodologies. Discrepancies emerge, necessitating the individual determination of the error factor. Remarkably, when utilizing the generalized proportional difference operator and its inverse, the need for a separate determination of the error factor is obviated. The equivalence between numerical solutions and closed form solutions is consistently observed in our methodologies.

In this article, the generalized proportional delta operator $\Delta_\ell^{-\mu}$ has been proposed. Section 2 studies the generalized discrete proportional case of the Leibnitz theorem, binomial expansion, and Montmort's formulas by defining the generalized proportional difference operator. In Section 3, we establish the inverse generalized proportional delta operator and derives exact and numerical solutions of various functions. In Section 4, we discuss the applications in various types of arithmetic progression (AP) of finite series involving the operator $\Delta_\ell^{-\mu}$. In Section 5, we define the generalized proportional discrete Laplace transform (GDPLT) and obtain the GDPLT of various functions. To demonstrate our findings, appropriate examples are supplied and confirmed using MATLAB in Section 6. Finally, the conclusion of this paper is given in Section 7.

2. Basic Definitions and Results

Prior to presenting and proving our results, we lay the foundation by introducing essential definitions and lemmas. This section unveils fundamental concepts and provides preliminary results that will prove crucial in our subsequent discussions.

Definition 1 ([27]). *A Conformable Differential Operator is defined by*

$$D^\alpha f(t) = \alpha f'(t) + (1 - \alpha)f(t), \quad \alpha \in [0, 1], \tag{1}$$

which represents the α-derivative of $f(t)$. Here, D^α is conformable provided the function $f(t)$ is differentiable at t and $f'(t) = \frac{d}{dt}f(t)$.

Definition 2 ([23]). *Let $f(t)$ be real- or complex-valued function on $[0, \infty)$. Then, the forward (α, β)-difference operator $\Delta_{(\alpha,\beta)(\ell)}$ on $f(t)$ is defined by*

$$\ell\Delta_{(\alpha,\beta)(\ell)}f(t) = \beta f(t + \ell) - \alpha f(t), \quad \ell \in (0, \infty). \tag{2}$$

Remark 1 ([19,23]). *When $\alpha = 1$ and $\beta = 1$, the difference operator $\Delta_{(1,1)(\ell)}$ becomes the generalized difference operator $\ell\Delta_\ell$, defined as $\ell\Delta_\ell f(t) = f(t + \ell) - f(t)$.*

Definition 3. *By Definition 1, the generalized proportional delta operator for the function $w(t) : [0, \infty) \to \mathbb{C}, \ell > 0$ is defined by*

$$\Delta_\ell^\mu w(t) = \mu \Delta_\ell w(t) + (1 - \mu)w(t), \ \mu \in [0, 1]. \tag{3}$$

Lemma 1. *If $L^\ell w(t) = w(t + \ell)$ is the usual lead operator, then*

$$L^\ell = \frac{1}{\mu}(\Delta_\ell^\mu + 2\mu - 1) = (1 + \Delta)^\ell = (1 + \Delta_\ell). \tag{4}$$

Proof. From the usual lead operator L^ℓ, we write

$$L^\ell w(t) = w(t + \ell) = w(t + \ell) - w(t) + w(t) = (1 + \Delta_\ell)w(t). \tag{5}$$

When $\ell = 1$, (5) becomes $L = \Delta + 1$. Then, from (3) and (5), we obtain

$$(1 + \Delta_\ell)w(t) = L^\ell w(t) = (1 + \Delta)^\ell w(t). \tag{6}$$

$$L^\ell w(t) = \frac{1}{\mu}(\Delta_\ell^\mu + 2\mu - 1)w(t). \tag{7}$$

By comparing Equations (6) and (7), we complete the proof. $\square$

Lemma 2. *If a, b are constants and $w_1(t)$, $w_2(t)$ are any real- or complex-valued functions on $t \in [0, \infty)$, then*

(i) $\Delta_\ell^\mu[aw_1(t) + bw_2(t)] = a\Delta_\ell^\mu w_1(t) + b\Delta_\ell^\mu w_2(t)$;

(ii) $\Delta_\ell^{\mu_1}\Delta_\ell^{\mu_2}w(t) = \Delta_\ell^{\mu_2}\Delta_\ell^{\mu_1}w(t) = \Delta_\ell^{\mu_1+\mu_2}w(t)$;

(iii) $\Delta_\ell^\mu[w_1(t)w_2(t)] = \Delta_\ell^\mu w_2(t)L^\ell w_1(t) + (2\mu - 1)w_2(t)\Delta_\ell w_1(t)$;

(iv) $\Delta_\ell^\mu\left(\frac{w_1(t)}{w_2(t)}\right) = \dfrac{w_2(t)\Delta_\ell^\mu w_1(t) - (2\mu - 1)w_1(t)\Delta_\ell w_2(t)}{w_2(t)L^\ell w_2(t)}.$

Proof. The proof follows by Definition 3. $\square$

Definition 4. *By Definition 3, the second order of Δ_ℓ^μ is defined as $\Delta_\ell^{2\mu} = \Delta_\ell^\mu\Delta_\ell^\mu$ and, in general, the nth order of Δ_ℓ^μ is defined as $\Delta_\ell^{n\mu} = \Delta_\ell^{(n-1)\mu}(\Delta_\ell^\mu)$.*

The following are easy deductions.

(i)   For the positive integers p, q and $\mu \in [0, 1]$, $\Delta_\ell^{p\mu}\Delta_\ell^{q\mu} = \Delta_\ell^{q\mu}\Delta_\ell^{p\mu}$;

(ii) For $\mu \in [0, 1]$ and $w : [0, \infty) \to \mathbb{C}$ $\Delta_\ell^{n\mu}(cw(t)) = c\Delta_\ell^{n\mu}w(t)$;

(iii) $L^\ell \Delta_\ell^\mu w(t) = \Delta_\ell^\mu L^\ell w(t)$;

(iv) For the function $w : [0, \infty) \to \mathbb{C}$,

$$\Delta_\ell^{r\mu} = \sum_{i=0}^{r}(1 - 2\mu)^i rC_i(\mu L)^{\ell(r-i)} \tag{8}$$

and hence, $\Delta_\ell^{r\mu}w(t) = \sum_{i=0}^{r}(1 - 2\mu)^i rC_i \mu^{r-i}w(t + (r - i)\ell)$;

(v) $\Delta_{n\ell}^\mu = \mu L^{n\ell} + (1 - 2\mu)$;

(vi) $\Delta_{n\ell}^{\mu} = \frac{1}{\mu^{n-1}}[\Delta_{\ell}^{\mu} + (2\mu - 1)]^{n} + (1 - 2\mu);$

(vii) $\Delta_{n\ell}^{\mu} = \frac{1}{\mu^{n-1}} \sum\limits_{k=0}^{r} rC_k \Delta_{\ell}^{(r-k)\mu}(2\mu - 1)^{k} + (1 - 2\mu).$

The following theorem presents the generalized discrete proportional case of Leibniz's theorem.

Theorem 1. *Let $w(t)$ and $z(t)$, $t \in [0, \infty)$, be any two functions. Then,*

$$\Delta_{\ell}^{n\mu}[w(t)z(t)] = \sum\limits_{r=0}^{n} nC_r\left(2 - \frac{1}{\mu}\right)^{n-r} \sum\limits_{i=0}^{r}\left(\frac{2\mu^2 - 3\mu + 1}{\mu}\right)^{i} rC_i \times$$
$$\Delta_{\ell}^{(r-i)\mu}w(t)\Delta_{\ell}^{(n-r)\mu}z(t + (r-i)\ell), \ \mu \in [0, 1].$$

Proof. Define the operators L_1^{ℓ}, L_2^{ℓ}, $(\Delta_{\ell}^{n\mu})_1$, $(\Delta_{\ell}^{n\mu})_1$ and hence L^{ℓ} as

$$L_1^{\ell}w(t)z(t) = w(t + \ell)z(t), \ L_2^{\ell}w(t)z(t) = w(t)z(t + \ell), \tag{9}$$

$$(\Delta_{\ell}^{n\mu})_1 = \mu L_1^{\ell} + (1 - 2\mu), \ (\Delta_{\ell}^{n\mu})_2 = \mu L_2^{\ell} + (1 - 2\mu), \ L^{\ell} = L_1^{\ell}L_2^{\ell}. \tag{10}$$

From (4) and (10), we have $\Delta_{\ell}^{\mu} = \mu L^{\ell} + 1 - 2\mu = \mu L_1^{\ell}L_2^{\ell} + 1 - 2\mu = \frac{2\mu - 1}{\mu}(\Delta_{\ell}^{\mu})_2 +$
$(\Delta_{\ell}^{\mu})_1 L_2^{\ell} + H$, where $H = \frac{2\mu^2 - 3\mu + 1}{\mu}$.

From the above expression, we obtain

$$\Delta_{\ell}^{n\mu}(w(t)z(t)) = \left[(2 - \mu^{-1})(\Delta_{\ell}^{\mu})_2 + (\Delta_{\ell}^{\mu})_1 L_2^{\ell} + H\right]^{n}w(t)z(t). \tag{11}$$

The results follow from the binomial theorem, (9)–(11). □

Next, we establish the generalized discrete proportional form of the binomial expansion.

Theorem 2. *For $n, m \in \mathbb{Z}^{+}$, the generalized discrete proportional case of the binomial expansion is given by*

$$(t + n\ell)^{m} = \sum\limits_{p=0}^{n} nC_p \frac{1}{(2\mu - 1)^{p}} \sum\limits_{q=0}^{p} pC_q\mu^{p-q}(1 - 2\mu)^{q}[t + (p - q)\ell]^{m}. \tag{12}$$

Proof. From (4), we have $L^{\ell} = \frac{1}{\mu}(\Delta_{\ell}^{\mu} + 2\mu - 1)$. Furthermore, $L^{n\ell} = \left[\frac{1}{\mu}(\Delta_{\ell}^{\mu} + 2\mu - 1)\right]^{n}$. This implies

$$L^{n\ell}w(t) = \left[\frac{1}{\mu}(\Delta_{\ell}^{\mu} + 2\mu - 1)\right]^{n}w(t). \tag{13}$$

The proof follows by taking $w(t) = t^{m}$ on (13) and using binomial theorem. □

We now introduce the discrete case of Montmort's generalized proportional formula.

Lemma 3. *If the series $\sum\limits_{t=0}^{\infty} c_{k\ell}t^{k\ell}$ converges, then*

$$\sum\limits_{t=0}^{\infty} c_{k\ell}t^{k\ell} = \mu \sum\limits_{t=0}^{\infty} \frac{x^{t\ell}\Delta_{\ell}^{t\mu}}{(\mu - (2\mu - 1)x^{\ell})^{t+1}}c_0. \tag{14}$$

Proof. Using the lead operator $L^{\ell} = \frac{1}{\mu}(\Delta_{\ell}^{\mu} + 2\mu - 1)$, we can get

$\sum\limits_{t=0}^{\infty} c_{t\ell}x^{t\ell} = \sum\limits_{t=0}^{\infty} x^{t\ell}L^{t\ell}c_0 = \{1 - x^{\ell}L^{\ell}\}^{-1}c_0 = \{1 - x^{\ell}\frac{1}{\mu}(\Delta_{\ell}^{\mu} + 2\mu - 1)\}^{-1}c_0.$

Now, the proof follows by using the binomial expansion. $\quad\square$

3. Inverse of the Generalized Proportional Difference Operator

In this section, we establish the inverse of the generalized proportional difference operator $\Delta_\ell^{-\mu}$ and present explicit formulas for partial sums involving higher powers of the geometric arithmetic progression. Additionally, we illustrate the application of these formulas through relevant examples, providing a comprehensive understanding of their utility in practical scenarios.

Definition 5. *By Definition 3, we define the inverse of the generalized proportional difference operator represented as follows. If $\Delta_\ell^\mu z(t) = w(t)$, then*

$$\Delta_\ell^{-\mu} w(t) = z(t) - \left(2 - \mu^{-1}\right)^{\left[\frac{t}{\ell}\right]} z(j), \tag{15}$$

where $z(j)$ is constant and the nth order inverse generalized proportional difference operator represented by $\Delta_\ell^{-n\mu}$ is defined as

$$\Delta_\ell^{-n\mu} w(t) = \Delta_\ell^{-(n-1)\mu} \Delta_\ell^{-\mu} w(t).$$

Lemma 4. *Let $\mu \in [0,1)$, $w(t)=1$, $j = t - \left[\frac{t}{\ell}\right]\ell$ and $t \in [\ell, \infty)$. Then,*

$$\Delta_\ell^{-\mu}(1) = \frac{1}{(1-\mu)}\left[1 - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} j\right] = \frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1}. \tag{16}$$

Proof. From (3) and (15), we get the proof. $\quad\square$

Lemma 5. *If $\mu \in [0,1)$, ℓ is positive real and $t \in [0,\infty)$, then*

$$\Delta_\ell^{-\mu} w(t)\big|_j^t = \frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} w(t - k\ell). \tag{17}$$

Proof. The proof is followed by (15) and the relation

$$\Delta_\ell^\mu \left[\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} w(t - k\ell)\right] + c_j = w(t). \quad\square$$

The operator methods of summation on $\Delta_\ell^{-\mu}$ are as follows.

Theorem 3. *Assume that $P(t)$, $t \geq 2\ell$ is any function of t and $\lambda \neq 1$. Then, we obtain,*

$$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} L^{-\ell}(\lambda^k P^k) = \frac{\lambda^t}{(\lambda^\ell - 1)(1 - 2\mu)} \sum_{i=0}^{\infty} \frac{(-1)^i \lambda^{i\ell}}{(\lambda^\ell - 1)^i (1 - 2\mu)^i} P(t)\Big|_j^t, \tag{18}$$

where $L^{-\ell}(\lambda^k P^k) = \lambda^{t-k\ell} P(t - k\ell)$.

Proof. For a function $F(t)$, we find

$$\Delta_\ell^\mu \lambda^t F(t) = \lambda^t [\mu \lambda^\ell L^\ell + (1 - 2\mu)] F(t) = \lambda^t P(t), \tag{19}$$

hence, we obtain

$$[\mu \lambda^\ell L^\ell + (1 - 2\mu)]^{-1} P(t) = F(t). \tag{20}$$

From (19) and (15), we find

$$\Delta_\ell^{-\mu} \lambda^t P(t) = \lambda^t F(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} \lambda^j F(j),$$

and hence by (20), we obtain

$$\Delta_\ell^{-\mu} \lambda^t P(t) = \lambda^t [\mu \lambda^\ell L^\ell + (1 - 2\mu)]^{-1} P(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} c_j, \tag{21}$$

where $c_j = \lambda^j [\mu \lambda^\ell L^\ell + (1 - 2\mu)]^{-1} P(j)$. Equations (17) and (21), along with the binomial theorem, are now used to support the proof. □

4. Applications in Various Sorts of A.P. Finite Series Involving $\Delta_\ell^{-\mu}$

Utilizing $\Delta_\ell^{-\mu}$ as a tool, we derive expressions for the sums of both arithmetico-geometric progressions and arithmetico-double geometric progressions in the realm of number theory.

The following theorem presents the general formula for the summation of higher powers in an arithmetico-geometric progression.

Theorem 4. *Let $t \in [\ell, \infty)$, $\ell \in [0, \infty)$, $\mu \in [0, 1)$ and $j = t - \left[\frac{t}{\ell}\right]\ell$. Then,*

$$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} (t - k\ell)^m = F_n(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} F_n(j), \tag{22}$$

where $F_n(t) = \dfrac{t^m}{1 - \mu} + \displaystyle\sum_{k=1}^{m} \dfrac{(-\mu)^k}{(1 - \mu)^{k+1}} \Delta_\ell^\mu t^m$.

Proof. From (16), we have

$$\Delta_\ell^{-\mu}(1) = F_0(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} F_0(j), \tag{23}$$

where $F_0(t) = \dfrac{1}{(1 - \mu)}$. Since $\Delta_\ell^\mu t = (1 - \mu)t + \mu\ell$, (15) and (23) give

$$\Delta_\ell^{-\mu} t = F_1(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} F_1(j), \tag{24}$$

$F_1(t) = \dfrac{t}{(1 - \mu)^2} - \dfrac{\mu\ell}{(1 - \mu)^2}$. Since $\Delta_\ell^\mu t^2 = (1 - \mu)t^2 + 2\mu\ell t + \mu\ell^2$, Δ_ℓ^μ is linear, (15), (23) and (24) yield

$$\Delta_\ell^{-\mu} t^2 = F_2(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} F_2(j), \tag{25}$$

where $F_2(t) = \dfrac{t^2}{(1 - \mu)} + \displaystyle\sum_{k=1}^{2} \dfrac{(-\mu)^k}{(1 - \mu)^{k+1}} \Delta_\ell^k t^2$. Similarly,

$$\Delta_\ell^\mu t^3 = (1 - \mu)t^3 + 3\mu\ell t^2 + 3\mu\ell^2 t + 2\mu\ell^3, \tag{26}$$

and from (15) and (26), we find

$$\Delta_\ell^{-\mu} t^3 = F_3(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} F_3(j), \tag{27}$$

where $F_3(t) = \dfrac{t^3}{(1-\mu)} + \sum\limits_{k=1}^{3} \dfrac{(-\mu)^k}{(1-\mu)^{k+1}} \Delta_\ell^\mu t^3$. Now, proceeding with the same process for $\Delta_\ell^{-\mu} t^m$, we get the proof. $\square$

Theorem 5. *Let $t \in [\ell, \infty)$, $\ell \in [0, \infty)$, $\mu \in [0, 1)$ and $j = t - \left[\frac{t}{\ell}\right]\ell$. Then,*

$$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} (t - k\ell)_\ell^{(m)} = G_n(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} G_n(j), \tag{28}$$

where $G_n(t) = \dfrac{t^{(m)}}{1-\mu} + \sum\limits_{k=1}^{m} \dfrac{(-\mu)^k}{(1-\mu)^{k+1}} \Delta_\ell^\mu t^{(m)}$.

Proof. From (24), we have

$$\Delta_\ell^{-\mu} t_\ell^{(1)} = G_1(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} G_1(j), \tag{29}$$

where $G_1(t) = \dfrac{t_\ell^{(1)}}{(1-\mu)} - \dfrac{\mu\ell}{(1-\mu)^2}$. Since $\Delta_\ell^\mu t_\ell^{(2)} = (1-\mu)t_\ell^{(2)} + 2\mu\ell t_\ell^{(1)}$, Δ_ℓ^μ is linear, (15) and (29) yield

$$\Delta_\ell^{-\mu} t_\ell^{(1)} = G_2(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} G_2(j), \tag{30}$$

where $G_2(t) = \dfrac{t_\ell^{(2)}}{(1-\mu)} + \sum\limits_{k=1}^{2} \dfrac{(-\mu)^k}{(1-\mu)^{k+1}} \Delta_\ell^\mu t_\ell^{(1)}$. Similarly, since

$$\Delta_\ell^\mu t_\ell^{(3)} = (1-\mu)t_\ell^{(3)} + 3\mu\ell t^2 - \mu\ell^2 t, \tag{31}$$

from Definitions (15) and (31), we find

$$\Delta_\ell^{-\mu} t_\ell^{(3)} = G_3(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} G_3(j), \tag{32}$$

where $G_3(t) = \dfrac{t_\ell^{(3)}}{(1-\mu)} + \sum\limits_{k=1}^{3} \dfrac{(-\mu)^k}{(1-\mu)^{k+1}} \Delta_\ell^\mu t_\ell^{(3)}$. Now, proceeding with the same process for $\Delta_\ell^{-\mu} t_\ell^{(m)}$, we get the proof. $\square$

The following theorem provides the formula for the sum of an arithmetico-double geometric progression.

Theorem 6. *If $t \in [\ell, \infty)$ and $j = t - \left[\frac{t}{\ell}\right]$, then*

$$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} (t - k\ell)^m a^{t-k\ell} = H_m(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} H_m(j), \tag{33}$$

where

$$H_m(t) = \frac{t^m a^t}{1 - 2\mu + \mu a^\ell} + \sum_{k=1}^{m} \frac{(-\mu)^k}{(1 - 2\mu + \mu a^\ell)^{k+1}} \Delta_\ell(t^m) L^{m\ell}(a^t).$$

Proof. $\Delta_\ell^\mu a^t = (1 - 2\mu + \mu a^\ell)a^t$, which yields from (15) that

$$\Delta_\ell^{-\mu} a^t = \frac{a^t}{1 - 2\mu + \mu a^\ell} - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} \frac{a^j}{1 - 2\mu + \mu a^\ell}. \tag{34}$$

$\Delta_\ell^\mu t a^t = (1 - 2\mu + \mu a^\ell) t a^t + \mu \ell a^\ell a^t$, which yields from (15) that

$$\Delta_\ell^{-\mu}(t a^t) = \left[\frac{t a^t}{1 - 2\mu + \mu a^\ell} - \frac{\mu \ell a^\ell}{(1 - 2\mu + \mu a^\ell)^2} a^t \right]_j^t. \tag{35}$$

$\Delta_\ell^\mu t^2 a^t = (1 - 2\mu + \mu a^\ell) t^2 a^t + 2\mu \ell a^\ell t a^t + \mu \ell^2 a^\ell a^t$, which yields from (15) that

$$\Delta_\ell^{-\mu}(t^2 a^t) = \left[\frac{t^2 a^t}{1 - 2\mu + \mu a^\ell} + \sum_{k=1}^{2} \frac{(-\mu)^k}{(1 - 2\mu + \mu a^\ell)^3} \Delta_\ell(t^m) L^{2\ell} a^t \right]_j^t. \tag{36}$$

Now, we proceed with the same process for $w(t) = t^m a^t$, and this completes the proof. □

Remark 2. *The following Table 1 is a comparison of Theorem 3, Theorem 4, Theorem 5 and Theorem 6.*

Table 1. Comparison.

Functions	Numerical Solution	Exact Solution
1	$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1}$	$\frac{1}{(1-\mu)} \left[1 - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} j \right]$
t	$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} (t - k\ell)$	$F_1(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} F_1(j)$, where $F_1(t) = \frac{t}{(1-\mu)^2} - \frac{\mu \ell}{(1-\mu)^2}$
t^m	$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} (t - k\ell)^m$	$F_n(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} F_n(j)$, where $F_n(t) = \frac{t^m}{1-\mu} + \sum_{k=1}^{m} \frac{(-\mu)^k}{(1-\mu)^{k+1}} \Delta_\ell^\mu t^m$
$t^{(m)}$	$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} (t - k\ell)^{(m)}$	$G_n(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} G_n(j)$, where $G_n(t) = \frac{t^{(m)}}{1-\mu} + \sum_{k=1}^{m} \frac{(-\mu)^k}{(1-\mu)^{k+1}} \Delta_\ell^\mu t^{(m)}$
$t^m a^t$	$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} (t - k\ell)^m a^{t-k\ell}$	$H_m(t) - \left(2 - \frac{1}{\mu}\right)^{\left[\frac{t}{\ell}\right]} H_m(j)$, where $H_m(t) = \frac{t^m a^t}{1 - 2\mu + \mu a^\ell} +$ $\sum_{k=1}^{m} \frac{(-\mu)^k}{(1 - 2\mu + \mu a^\ell)^{k+1}} \Delta_\ell(t^m) L^{m\ell}(a^t).$
$\lambda^k P^k$	$\frac{1}{\mu} \sum_{k=1}^{\left[\frac{t}{\ell}\right]} \left(2 - \frac{1}{\mu}\right)^{k-1} L^{-\ell}(\lambda^k P^k)$	$\frac{\lambda^t}{(\lambda^\ell - 1)(1 - 2\mu)} \sum_{i=0}^{\infty} \frac{(-1)^i \lambda^{i\ell}}{(\lambda^\ell - 1)^i (1 - 2\mu)^i} P(t) \Big|_j^t$

Remark 3. *Researchers can analyze discrete generalized proportional differences and inverse differences across various functions. Readers may apply this operator in Newton's law of cooling to reduce errors.*

5. Applications of the Generalized Discrete Proportional Laplace Transform Using $\Delta_\ell^{-\mu}$

Within this section, we delve into the application of $\Delta_\ell^{-\mu}$ to determine the GDPLT. The subsequent definition elucidates the GDPLT.

Definition 6. *By Definition 3, let $w(t)$ be a function of t defined for $t \geq 0$ and let $\ell \in [0,1]$. Then, the generalized discrete proportional Laplace transform is defined by*

$$\mathscr{L}_\mu \{w(t)\} = \mu \ell \Delta_\ell^{-\mu} e^{-st} w(t) \Big|_0^\infty + (1 - \mu) \Delta_\ell^{-\mu} t e^{-st} w(t) \Big|_0^\infty, \tag{37}$$

$$= \ell \sum_{t=1}^{\infty} (2 - \mu^{-1})^{r-1} T + \frac{1-\mu}{\mu} \sum_{t=1}^{\infty} (2 - \mu^{-1})^{r-1} (t - r\ell) T, \tag{38}$$

where $T = e^{-s(t-r\ell)} w(t - r\ell)$, and $\mathscr{L}_{\mu}\{w(t)\} = \mathscr{W}(s)$ and $\mathscr{L}_{\mu}^{-1}\{\mathscr{W}(s)\} = w(t)$ denote the inverse of the generalized discrete proportional Laplace transform.

Lemma 6 (Numerical solution). *Let $w(t)$ be a function of t defined for $t \in [0, \infty)$ and $\lim\limits_{t \to \infty} \Delta_{\ell}^{-\mu} w(t) = 0$. Then,*

$$\Delta_{\ell}^{-\mu} w(t)\big|_{t}^{\infty} = - \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(2\mu - 1)^k} w(t + k\ell). \tag{39}$$

Proof. The proof follows from the relation

$$\Delta_{\ell}^{\mu} \left[- \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(2\mu-1)^k} w(t + k\ell) \right] = w(t), \ w(j) = 0 \ \text{as} \ t \to \infty \ \text{and (3)}. \quad \square$$

Lemma 7 (Exact solution). *Let $w(t)$ be a function of $t \in (0, \infty)$, $s > 0$ and $\ell > 0$. Then, the generalized discrete proportional Laplace transform of $w(t)$ is given by*

$$\mathscr{L}_{\mu}\{w(t)\} = - \sum_{k=1}^{\infty} (2 - \mu^{-1})^{-k} \left[\ell + \frac{(1-\mu)\ell}{\mu} k \right] e^{-sk\ell} w(k\ell). \tag{40}$$

Proof. The generalized discrete proportional Laplace transform of $w(t)$ is

$$\begin{aligned} \mathscr{L}_{\mu}\{w(t)\} &= \mu \ell \Delta_{\ell}^{-\mu} e^{-sk\ell} w(k\ell)\big|_{0}^{\infty} + (1 - \mu) \Delta_{\ell}^{-\mu} k e^{-sk\ell} w(k\ell)\big|_{0}^{\infty} \\ &= \ell \sum_{k=1}^{\infty} (2 - \mu^{-1})^{-k} e^{-s(t+k\ell)} w(t + k\ell)\big|_{0}^{\infty} \\ &\quad + \frac{1-\mu}{\mu} \sum_{k=1}^{\infty} (2 - \mu^{-1})^{-k} (t + e\ell) e^{-s(t+k\ell)} w(t + k\ell)\big|_{0}^{\infty} \end{aligned}$$

Hence,

$$\mathscr{L}_{\mu}\{w(t)\} = - \sum_{k=1}^{\infty} (2 - \mu^{-1})^{-k} \left[\ell + \frac{(1-\mu)\ell}{\mu} k \right] e^{-sk\ell} w(k\ell). \tag{41}$$

The proof is complete. $\quad \square$

The following Proposition 1 establishes the equivalence between the numerical solution and the exact solution of the generalized discrete proportional Laplace transform.

Proposition 1. *Let $w(t)$ be a function of $t \in (0, \infty)$, $s > 0$ and $\ell > 0$. Then, the numerical and exact solutions of the generalized discrete proportional Laplace transform of $w(t)$ are equal, which is given by*

$$\sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(2\mu - 1)^k} w(t + k\ell) = \sum_{k=1}^{\infty} (2 - \mu^{-1})^{-k} \left[\ell + \frac{(1-\mu)\ell}{\mu} k \right] e^{-sk\ell} w(k\ell). \tag{42}$$

Proof. Comparing (39) and (41), we get the proof. $\quad \square$

Lemma 8. *Let $t \in [\ell, \infty)$ and $w(t) = e^{-st}$, $t > 0$. Then,*

$$\Delta_{\ell}^{-\mu}(e^{-st})\big|_{j}^{t} = \frac{e^{-st}}{\mu e^{-s\ell} - 2\mu + 1}\bigg|_{j}^{t}. \tag{43}$$

Proof. The proof follows by (3) and (15). □

Lemma 9. *Let* $t \in [\ell, \infty)$ *and* $w(t) = te^{-st}$, $t > 0$. *Then,*

$$\Delta_\ell^{-\mu}(te^{-st})\Big|_j^t = \frac{te^{-st}}{\mu e^{-s\ell} - 2\mu + 1}\Big|_j^t - \frac{\mu\ell e^{-s(t+\ell)}}{(\mu e^{-s\ell} - 2\mu + 1)^2}\Big|_j^t. \tag{44}$$

Proof. Using (3), (15) and Lemma 8, the proof is complete. □

Proposition 2. *Let* $w(t)$ *be a unit step function of* t *and* $\ell > 0$. *Then, the generalized discrete proportional Laplace transform of* $w(t)$ *is given by*

$$\mathscr{L}_\mu\{w(t)\} = \frac{-\mu\ell}{\mu e^{-s\ell} - 2\mu + 1} + (1-\mu)\frac{\mu\ell e^{-s\ell}}{(\mu e^{-s\ell} - 2\mu + 1)^2}. \tag{45}$$

Proof. Taking $w(t) = 1$, $t > 0$ in (37), we get

$$\mathscr{L}_\mu\{1\} = \mu\ell\Delta_\ell^{-\mu}(e^{-st})\Big|_0^\infty + (1-\mu)\Delta_\ell^{-\mu}(te^{-st})\Big|_0^\infty. \tag{46}$$

Using (43) and (44) in (46), we get the proof. □

Lemma 10. *Let* $t \in [\ell, \infty)$ *and* $w(t) = e^{-(s-a)t}$, $t > 0$. *Then,*

$$\Delta_\ell^{-\mu}(e^{-(s-a)t})\Big|_j^t = \frac{e^{-(s-a)t}}{\mu e^{-(s-a)\ell} - 2\mu + 1}\Big|_j^t. \tag{47}$$

Proof. The proof follows by (3) and (15). □

Lemma 11. *Let* $t \in [\ell, \infty)$ *and* $w(t) = te^{-(s-a)t}$, $t > 0$. *Then,*

$$\Delta_\ell^{-\mu}(te^{-(s-a)t})\Big|_j^t = \frac{te^{-(s-a)t}}{\mu e^{-(s-a)\ell} - 2\mu + 1}\Big|_j^t - \frac{\mu\ell e^{-(s-a)(t+\ell)}}{(\mu e^{-(s-a)\ell} - 2\mu + 1)^2}\Big|_j^t. \tag{48}$$

Proof. Using (3), (15) and Lemma 8, the proof is complete. □

Proposition 3. *Let* $w(t) = e^{at}$ *be a function of* t *and* $\ell > 0$. *Then, the generalized discrete proportional Laplace transform of* $w(t)$ *is given by*

$$\mathscr{L}_\mu\{w(t)\} = \frac{-\mu\ell}{\mu e^{-(s-a)\ell} - 2\mu + 1} + (1-\mu)\frac{\mu\ell e^{-(s-a)\ell}}{(\mu e^{-(s-a)\ell} - 2\mu + 1)^2}. \tag{49}$$

Proof. Taking $w(t) = e^{at}$, $t > 0$ in (37), we get

$$\mathscr{L}_\mu\{e^{at}\} = \mu\ell\Delta_\ell^{-\mu}(e^{-(s-a)t})\Big|_0^\infty + (1-\mu)\Delta_\ell^{-\mu}(te^{-(s-a)t})\Big|_0^\infty. \tag{50}$$

Using (47) and (48) in (50), we get the proof. □

Lemma 12. *Let* $t \in [\ell, \infty)$ *and* $w(t) = e^{-(s+a)t}$, $a, t > 0$. *Then,*

$$\Delta_\ell^{-\mu}(e^{-(s+a)t})\Big|_j^t = \frac{e^{-(s+a)t}}{\mu e^{-(s+a)\ell} - 2\mu + 1}\Big|_j^t. \tag{51}$$

Proof. The proof follows by (3) and (15). □

Lemma 13. *Let* $t \in [\ell, \infty)$ *and* $w(t) = te^{-(s+a)t}$, $a, t > 0$. *Then,*

$$\Delta_\ell^{-\mu}(te^{-(s+a)t})\Big|_j^t = \frac{te^{-(s+a)t}}{\mu e^{-(s+a)\ell} - 2\mu + 1}\Big|_j^t - \frac{\mu\ell e^{-(s+a)(t+\ell)}}{(\mu e^{-(s+a)\ell} - 2\mu + 1)^2}\Big|_j^t. \tag{52}$$

Proof. Using (3), (15) and Lemma 8, the proof is complete. $\square$

Theorem 7. *Let* $w(t) = e^{-at}$ *be a function of* t *and* $a, \ell > 0$. *Then, the generalized discrete proportional Laplace transform of* $w(t)$ *is given by*

$$\mathscr{L}_\mu\{w(t)\} = \frac{-\mu\ell}{\mu e^{-(s+a)\ell} - 2\mu + 1} + (1 - \mu)\frac{\mu\ell e^{-(s+a)\ell}}{(\mu e^{-(s-a)\ell} - 2\mu + 1)^2}. \tag{53}$$

Proof. Taking $w(t) = e^{-at}$, $t > 0$ in (37), we get

$$\mathscr{L}_\mu\{e^{-at}\} = \mu\ell\Delta_\ell^{-\mu}(e^{-(s+a)t})\Big|_0^\infty + (1 - \mu)\Delta_\ell^{-\mu}(te^{-(s-a)t})\Big|_0^\infty. \tag{54}$$

Using (51) and (52) in (54), we get the proof. $\square$

Lemma 14. *Let* $t \in [\ell, \infty)$ *and* $w(t) = e^{-(s-ia)t}$, $a, t > 0$. *Then,*

$$\Delta_\ell^{-\mu}(e^{-(s-ia)t})\Big|_j^t = \frac{e^{-(s-ia)t}}{\mu e^{-(s-ia)\ell} - 2\mu + 1}\Big|_j^t. \tag{55}$$

Proof. The proof follows by (3) and (15). $\square$

Lemma 15. *Let* $t \in [\ell, \infty)$ *and* $w(t) = te^{-(s-ia)t}$, $a, t > 0$. *Then,*

$$\Delta_\ell^{-\mu}(te^{-(s-ia)t})\Big|_j^t = \frac{te^{-(s-ia)t}}{\mu e^{-(s-ia)\ell} - 2\mu + 1}\Big|_j^t - \frac{\mu\ell e^{-(s-ia)(t+\ell)}}{(\mu e^{-(s-ia)\ell} - 2\mu + 1)^2}\Big|_j^t. \tag{56}$$

Proof. Using (3), (15) and Lemma 8, the proof is complete. $\square$

Theorem 8. *Let* $w(t) = e^{iat}$ *be a function of* t *and* $a, \ell > 0$. *Then, the generalized discrete proportional Laplace transform of* $w(t)$ *is given by*

$$\mathscr{L}_\mu\{e^{iat}\} = \frac{-\mu\ell}{\mu e^{-(s-ia)\ell} - 2\mu + 1} + (1 - \mu)\frac{\mu\ell e^{-(s-ia)\ell}}{(\mu e^{-(s-ia)\ell} - 2\mu + 1)^2}. \tag{57}$$

or

$$\mathscr{L}_\mu\{\cos at\} = -\mu\ell\frac{\mu e^{-s\ell}\cos a\ell - 2\mu + 1}{(\mu e^{-s\ell}\cos a\ell - 2\mu + 1)^2 + (\mu e^{-s\ell}\sin a\ell)^2} + (1 - \mu)\ell$$

$$\frac{\mu^3 e^{-3s\ell}\cos a\ell - 2\mu^2(2\mu - 1)e^{-2s\ell} + \mu(2\mu - 1)^2 e^{-s\ell}\cos a\ell}{(A - 2\mu(2\mu - 1)e^{-s\ell}\cos a\ell + (2\mu + 1)^2)^2 + (B - 2\mu(2\mu - 1)e^{-s\ell}\sin a\ell)^2}, \tag{58}$$

and

$$\mathscr{L}_\mu\{\sin at\} = -\mu\ell\frac{\mu e^{-s\ell}\sin a\ell}{(\mu e^{-s\ell}\cos a\ell - 2\mu + 1)^2 + (\mu e^{-s\ell}\sin a\ell)^2} + (1 - \mu)\ell$$

$$\frac{\mu^3 e^{-3s\ell}\sin a\ell + \mu(2\mu - 1)^2 e^{-s\ell}\sin a\ell}{(A - 2\mu(2\mu - 1)e^{-s\ell}\cos a\ell + (2\mu + 1)^2)^2 + (B - 2\mu(2\mu - 1)e^{-s\ell}\sin a\ell)^2}, \tag{59}$$

where $A = \mu^2 e^{-2s\ell}\cos a\ell$ *and* $B = \mu^2 e^{-2s\ell}\sin a\ell$.

Proof. Taking $w(t) = e^{iat}$, $t > 0$ in (37), we get

$$\mathscr{L}_\mu\{e^{iat}\} = \mu\ell\Delta_\ell^{-\mu}(e^{-(s-ia)t})\big|_0^\infty + (1-\mu)\Delta_\ell^{-\mu}(te^{-(s-ia)t})\big|_0^\infty. \tag{60}$$

Using (55) and (56) in (60), we get (57), and equating the coefficients of real and imaginary parts, we get (58) and (59). $\quad\square$

Lemma 16. *Let $t \in [\ell, \infty)$ and $w(t) = t^2 e^{-st}$, $t > 0$. Then,*

$$\Delta_\ell^{-\mu}(t^2 e^{-st})\big|_j^t = \frac{t^2 e^{-st}}{\mu e^{-s\ell} - 2\mu + 1}\Big|_j^t - \frac{\mu\ell e^{-s\ell}}{(\mu e^{-s\ell} - 2\mu + 1)^2}(2t + \mu e^{-s\ell} + 1)e^{-st}\big|_j^t. \tag{61}$$

Proof. Using (3), (15) and Lemma (8), the proof is complete. $\quad\square$

Proposition 4. *Let $w(t) = t$ be a function of t and $\ell > 0$. Then, the generalized discrete proportional Laplace transform of $w(t)$ is given by*

$$\mathscr{L}_\mu\{w(t)\} = \frac{-\mu^2\ell^2}{(\mu e^{-s\ell} - 2\mu + 1)^2} + (1-\mu)\frac{2\mu^2\ell^2 e^{-2s\ell} + \mu\ell^2 e^{-s\ell}}{(\mu e^{-s\ell} - 2\mu + 1)^2}. \tag{62}$$

Proof. Taking $w(t) = t$, $t > 0$ in (37), we get

$$\mathscr{L}_\mu\{t\} = \mu\ell\Delta_\ell^{-\mu}(te^{-st})\big|_0^\infty + (1-\mu)\Delta_\ell^{-\mu}(t^2 e^{-st})\big|_0^\infty. \tag{63}$$

Using (43) and (61) in (63), we get the proof. $\quad\square$

6. Numerical Examples

Within this section, we showcase examples and figures that demonstrate the application of the GDPD and the generalized discrete proportional Laplace transform to various functions.

Definition 3 is demonstrated using the following example.

Example 1. *Let $\mu \in [0,1]$ and $n \in \mathbb{Z}$. The generalized proportional difference of a function as follows:*

(i) $w(t) = t^2$ at $\mu = \frac{n}{5}$, $0 \le n \le 5$, we have the following:

$\mu = 0, \ell \in (0,\infty): \quad \Delta_\ell^0 t^2 = t^2$

$\mu = \frac{1}{5}, \ell \in (0,\infty): \quad \Delta_\ell^{\frac{1}{5}} t^2 = (0.2)(2t\ell + \ell^2) + (0.8)t^2$

$\mu = \frac{2}{5}, \ell \in (0,\infty): \quad \Delta_\ell^{\frac{2}{5}} t^2 = (0.4)(2t\ell + \ell^2) + (0.6)t^2$

$\mu = \frac{3}{5}, \ell \in (0,\infty): \quad \Delta_\ell^{\frac{3}{5}} t^2 = (0.6)(2t\ell + \ell^2) + (0.4)t^2$

$\mu = \frac{4}{5}, \ell \in (0,\infty): \quad \Delta_\ell^{\frac{4}{5}} t^2 = (0.8)(2t\ell + \ell^2) + (0.2)t^2$

$\mu = 1, \ell \in (0,\infty): \quad \Delta_\ell^1 t^2 = (2t\ell + \ell^2).$

(ii) $w(t) = log(at) + e^{at}$, $t \neq 0$ and $a = 0.5$ at $\mu = \frac{n}{5}$, $0 \le n \le 5$, we have the following:

$\mu = 0, \ell \in (0,\infty): \quad \Delta_\ell^0(log(at) + e^{at}) = e^{at}$

$\mu = \frac{1}{5}, \ell \in (0,\infty): \quad \Delta_\ell^{\frac{1}{5}}(log(at) + e^{at}) = (0.2)log\left(\frac{t+\ell}{t}\right) + (0.6 + 0.2a^\ell)e^{at}$

$\mu = \frac{2}{5}, \ell \in (0,\infty): \quad \Delta_\ell^{\frac{2}{5}}(log(at) + e^{at}) = (0.4)log\left(\frac{t+\ell}{t}\right) + (0.2 + 0.4a^\ell)e^{at}$

$\mu = \frac{3}{5}, \ell \in (0,\infty): \quad \Delta_\ell^{\frac{3}{5}}(log(at) + e^{at}) = (0.6)log\left(\frac{t+\ell}{t}\right) + (-0.2 + 0.6a^\ell)e^{at}$

$\mu = \frac{4}{5}, \ell \in (0,\infty): \quad \Delta_\ell^{\frac{4}{5}}(log(at) + e^{at}) = (0.8)log\left(\frac{t+\ell}{t}\right) + (-0.6 + 0.8a^\ell)e^{at}$

$\mu = 1, \ell \in (0,\infty): \quad \Delta_\ell^1(log(at) + e^{at}) = log\left(\frac{t+\ell}{t}\right) + (-1 + a^\ell)e^{at}.$

The following Figure 1a–d shows the graphical representation for the given fuction $w(t) = t^2$.
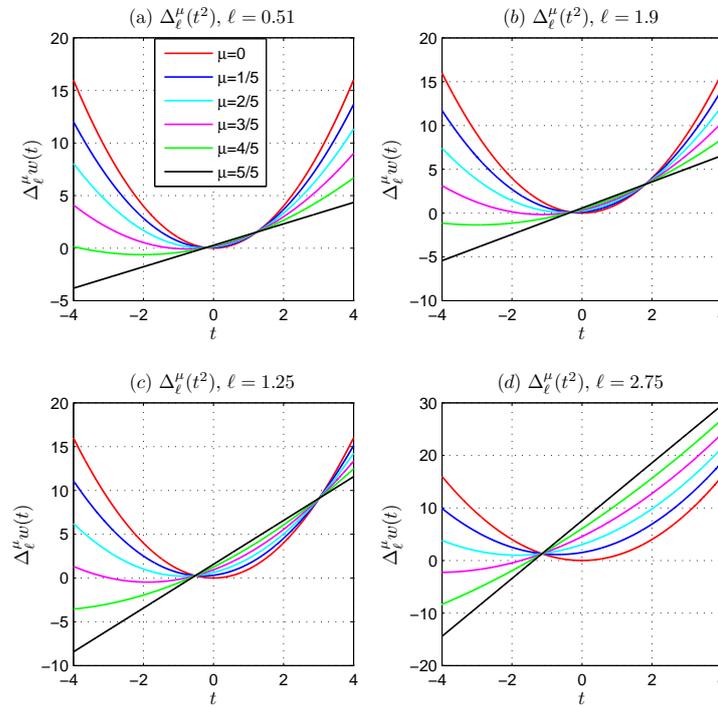
Figure 1. Generalized proportional difference of $w(t) = t^2$ for different shift values ℓ.

In Figure 1, we visually depict the transformation of the generalized proportional differences of t^2 from t^2 to $2t\ell + \ell^2$ as μ increases from 0 to 1. Moreover, the graphical representation in Figure 2 depicts the generalized proportional difference of $w(t) = \log(at) + e^{at}$ for various shift values ℓ.
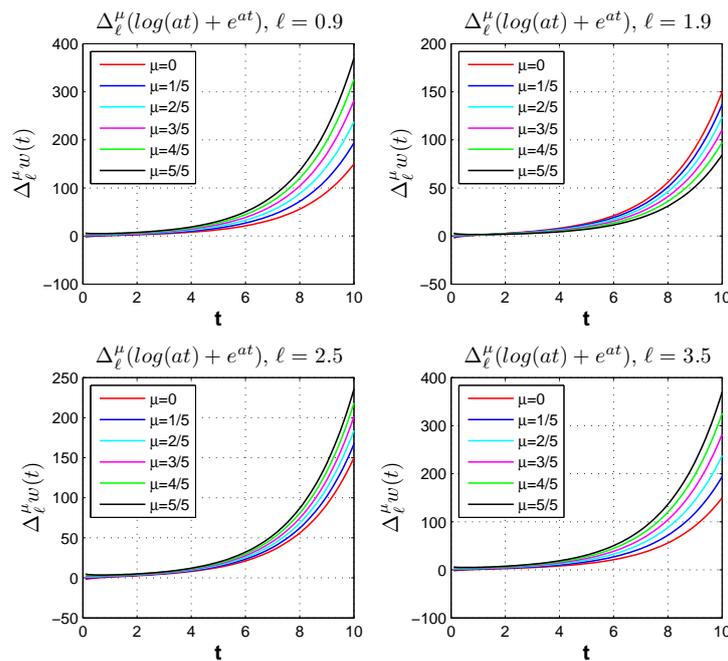
Figure 2. Generalized proportional difference of $w(t) = log(at) + e^{at}$ for different shift values ℓ.

The efficacy of Theorem 2 is demonstrated through the illustrative example presented in Example 2.

Example 2. *If θ is expressed in degrees with positive real values in an anticlockwise orientation, then*

$$\tan(t + n\theta) = \sum_{i=0}^{n} nC_i \frac{1}{(2\mu-1)^i} \sum_{r=0}^{i} iC_r \mu^{i-r} (1-2\mu)^r \tan[t + (i-r)\ell].$$

The effectiveness of Theorem 4 is verified through the following illustrative Example 3.

Example 3. *By taking $\ell = 3$, $\mu = 0.2$ and $t = 7$ in (16), we obtain*

$$\frac{1}{0.2} \sum_{k=1}^{2} \left(2 - \frac{1}{0.2}\right)^{k-1} = \frac{1}{(1-0.2)} - \left(2 - \frac{1}{0.2}\right)^2 \frac{1}{(1-0.2)} = -10.$$

The effectiveness of Theorem 4 is verified through the following illustrative example.

Example 4. *Using $m = 2$, $t = 7$, $\ell = 2$ and $\mu = 0.6$ in (22), we obtain*

$$\frac{1}{0.6} \sum_{k=1}^{3} \left(2 - \frac{1}{0.6}\right)^{k-1} (7 - 2k)^2 = F_2(7) - \left(2 - \frac{1}{0.6}\right)^3 F_2(1) = 46.851851, \tag{64}$$

where $F_2(7) = 47.5$ and $F_2(1) = 17.5$.

The effectiveness of Theorem 5 is verified through the following illustrative Example 5.

Example 5. *Using $m = 3$, $t = 13$, $\ell = 3$ and $\mu = 0.3$ in (5), we get*

$$\frac{1}{0.3} \sum_{k=1}^{4} \left(2 - \frac{1}{0.3}\right)^{k-1} (13 - 3k)^{(3)} = G_3(13) - \left(2 - \frac{1}{0.3}\right)^3 G_3(1) = 682.46914, \tag{65}$$

where $F_2(7) = 749.79592$ and $F_2(1) = 21.25781$.

The effectiveness of Theorem 6 is verified through the following illustrative Example 6.

Example 6. *Substituting $m = 1$, $t = 7$, $\ell = 2$, $a = 2$ and $\mu = 0.1$ in Theorem 6, we get*

$$\frac{1}{0.1} \sum_{k=1}^{4} \left(2 - \frac{1}{0.1}\right)^{k-1} (7 - 2k) 2^{(7-2k)} = H_1(7) - \left(2 - \frac{1}{0.1}\right)^3 H_1(1) = 960, \tag{66}$$

where $H_1(7) = 675.55556$ and $H_1(1) = 0.55556$.

The following Figures 3–7 show the generalized discrete proportional Laplace transform of a unit step function, the generalized discrete proportional Laplace transform of e^{at}, the generalized discrete proportional Laplace transform of e^{-at}, the generalized discrete proportional Laplace transform of $cos(at)$, and the generalized discrete proportional Laplace transform of $sin(at)$.
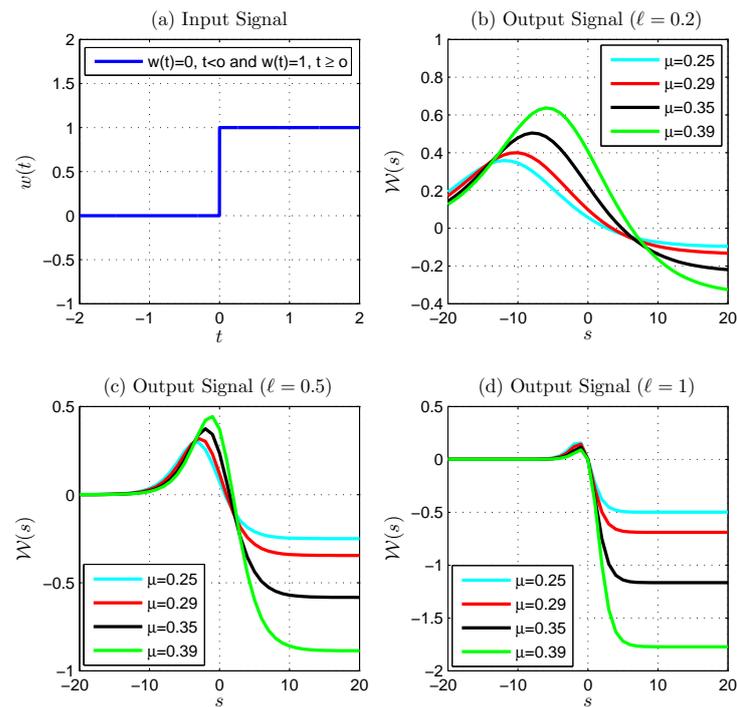
Figure 3. Generalized discrete proportional Laplace transform of the unit step function: (**a**) is the input signal, (**b**–**d**) show the output signal with fixed ℓ (0.2, 0.5, and 1) and various μ.
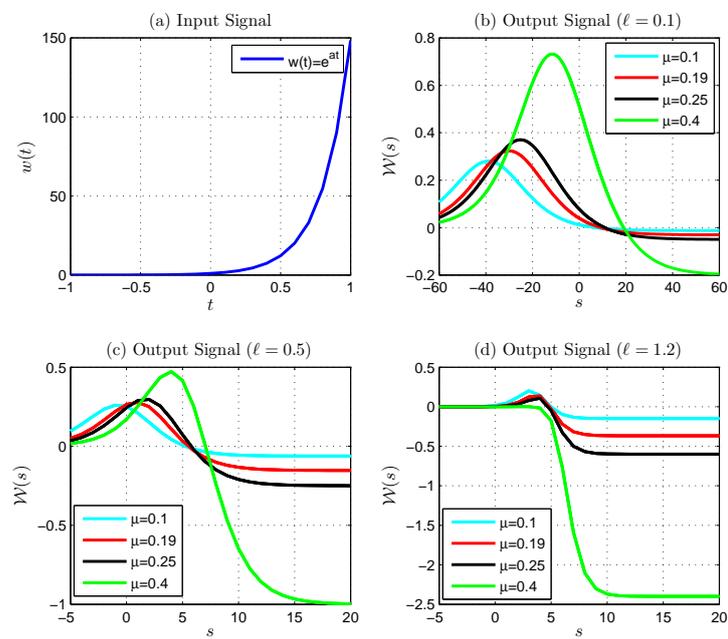
Figure 4. Generalized discrete proportional Laplace transform of e^{at}: (**a**) is the input signal, (**b**–**d**) shows the output signal with fixed ℓ (0.2, 0.5, and 1) and various μ.
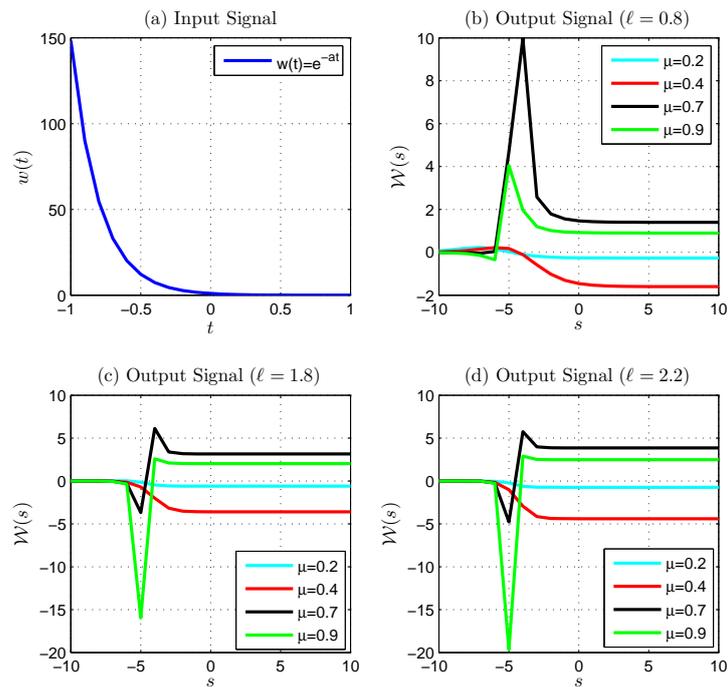
Figure 5. Generalized discrete proportional Laplace transform of e^{-at}: (**a**) is the input signal, (**b**–**d**) shows the output signal with fixed ℓ (0.2, 0.5, and 1) and various μ.
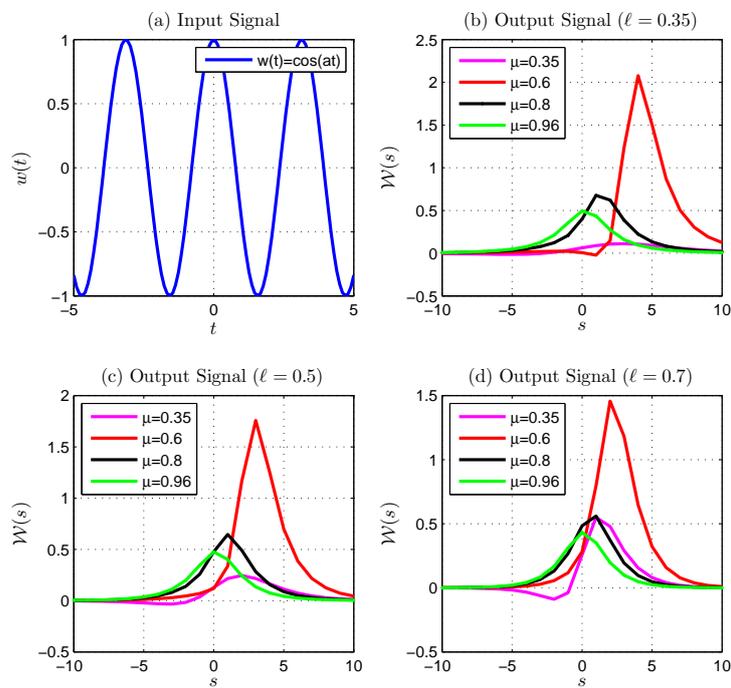
Figure 6. Generalized discrete proportional Laplace transform of $cos(at)$: (**a**) is the input signal, (**b**–**d**) shows the output signal with fixed ℓ (0.2, 0.5, and 1) and various μ.
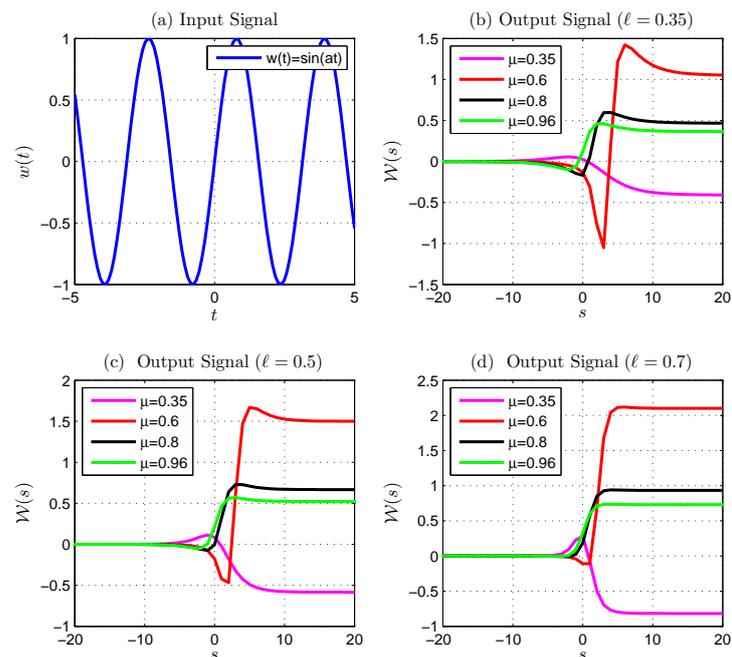
Figure 7. Generalized discrete proportional Laplace transform of $sin(at)$: (**a**) is the input signal, (**b**–**d**) shows the output signal with fixed ℓ (0.2, 0.5, and 1) and various μ.

7. Conclusions

This article explores the generalized discrete proportional case of Leibnitz's theorem, binomial expansion, and Montmort's equations by establishing the difference operator. The authors developed the inverse generalized proportional delta operator and obtained exact and numerical solutions for diverse functions. We also addressed applications in various types of APs of finite series involving the operator, and defined and obtained the GDPLT of various functions. We provide and confirm MATLAB examples to demonstrate our findings. Similarly, one can find the generalized discrete proportional Laplace transform of other functions. Readers (researchers) can discover several applications in the fields of control systems and engineering, as well as image processing. Our future studies will continue in this vein.

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